Local Theory of Holomorphic Foliations and Vector Fields

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Foreword

These notes constitutes a slightly informal introduction to what is often called the *theory of (singular) holomorphic foliations* with emphasis on its local aspects that may also be called *singularity theory of holomorphic foliations and/or vector fields.* Though the text is unpolished, we believe to have provided detailed proofs for all the statements given here. The material is roughly split into two parts, namely Chapters 1 and 2. Chapter 1 is primarily devoted to motivation for the theory of "(singular) holomorphic foliations" and in this direction it contains several examples of these foliations appearing in situations of interest. It also contains some general background from analytic geometry and topology that are often used in their study. This chapter is essentially of global nature so its purpose is indeed to motivate (and to provide some for) the study of globally defined foliations/differential equations rather than the study of their singularities that is the object of Chapter 2. Explanations for this discrepancy begin by saying that these notes are planned to be continued in the future with the inclusion of new chapters. Besides it is interesting to point out that the (local) study of singularities already entails a global perspective once the dimension of the ambient is at least 3. The simplest way to notice the existence of global foliations (possibly with rather complicate dynamics) hidden into the structure of a singularity of a vector field defined on, say, $(\mathbb{C}^3, 0)$, is to perform the blow-up of the singularity in question (cf. Section 1.4.1). In general, the blown-up foliation will induce a globally defined foliation on the resulting exceptional divisor (isomorphic to the complex projective plane in this case). Actually this foliation depends only on the first non-zero homogeneous component of the Taylor series of the mentioned vector field. Besides all foliations on the projective plane are recovered in this way, see "Example 3" in Section 1.3.

Chapter 2 is then devoted to the singularity theory for vector fields/foliations. Most of the chapter is taken up by the case of singularities defined on $(\mathbb{C}^2, 0)$ where the theory has reached a high level of development, with landmark progresses being represented by the papers of Mattei-Moussu and of Martinet-Ramis, [M-M], [Ma-R]. Much of this chapter is indeed devoted to present the results of these papers. Yet we have also included some classical and more recent material concerning singularities in higher dimensions.

In a sense these notes are vaguely reminiscent from a course the first author lectured at Stony Brook several years ago. He would like to thank the audience of his course and specially Andre Carvalho and Misha Lyubich for their interest. The project of writing these notes, however, would never be undertaken if it were not by the interest of Gisela Marino to whom we are most grateful. Gisela was the first to put into typing and fill in details of definitions and proofs for most of what is now "Chapter 2", cf. [Mr]. Unfortunately she did not wish to continue her study and left to us to complete the present material (as well as the planned subsequent chapters).

The authors

Contents

Chapter 1

Foundational Material

We are interested in understanding the behavior of first order ordinary differential equations in $\mathbb{C}^2$, where the "time" parameter is in $\mathbb{C}$.

To begin with, let us recall the main aspects of real ordinary differential equations. Complex Ordinary Differential Equations (ODEs) can then be obtained from real ones from a natural "complexification" procedure. It will be seen that complex ODEs are closely related to singular holomorphic foliations.

1.1 Real Ordinary Differential Equations

Let $X : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field given by $X(x,y) = (P(x,y), Q(x,y))$, where P and Q are polynomials. The ordinary differential equation associated to this field is:

$$\frac{d}{dt}(x(t), y(t)) = X(x(t), y(t)) .$$

It can also be regarded as the following system of ODEs

$$\begin{cases} \frac{d}{dt}x(t) = P(x(t), y(t)) , \\ \\ \frac{d}{dt}y(t) = Q(x(t), y(t)) . \end{cases} \tag{1.1}$$

Since the vector field is sufficiently smooth (C^∞, in fact) we may assert that given a $t_0 \in \mathbb{R}$ and $(x_0, y_0) \in \mathbb{R}^2$, there exists a unique solution $\phi(t) = (\phi_1(t), \phi_2(t))$ of (1.1) defined on a neighborhood of t_0, and verifying

$$\phi(t_0) = (\phi_1(t_0), \phi_2(t_0)) = (x_0, y_0) .$$

Furthermore, there exists the concept of extending a local solution to obtain a solution defined on a maximal domain, i.e, on the "largest possible interval". This means that for each $(x_0, y_0) \in \mathbb{R}^2$ there exists an interval $I(x_0, y_0)$ and a solution φ of (1.1) defined on $I(x_0, y_0)$ which satisfies the following conditions:

1. $\varphi(t_0) = (x_0, y_0)$.

2. If ψ defined on $I \subseteq \mathbb{R}$, $t_0 \in I$, is another solution of (1.1) verifying $\psi(t_0) = (x_0, y_0)$, then, $I \subset I(x_0, y_0)$ and, furthermore, $\varphi|_I = \psi$.

Let $\Omega \subseteq \mathbb{R} \times \mathbb{R}^2$ be the open set $\{(t, x_0, y_0) \in \mathbb{R}^3; t \in I(x_0, y_0)\}$. The *flow* associated to (1.1) is defined to be:

$$\begin{aligned} \Phi : \Omega &\rightarrow \mathbb{R}^2 \\ (t, x_0, y_0) &\mapsto \phi(t)\,, \end{aligned}$$

where $\phi(t)$ is the solution of (1.1) such that $\phi(t_0) = (x_0, y_0)$.

Notice that the flow may not be *complete*. That is, it may not be defined for all $t \in \mathbb{R}$, since $I(x_0, y_0)$ is not necessarily the whole real line. On the other hand, it is easy to check that, when $I(x_0, y_0) \neq \mathbb{R}$ then $\Phi(t, x_0, y_0)$ tends to infinity as t approaches the extremities of $I(x_0, y_0)$, for a fixed (x_0, y_0). Here we say that $\Phi(t, x_0, y_0)$ "tends to infinity" in the sense that it leaves every compact set contained in the domain of definition of X.

Remark 1.1 The above discussion actually holds for every regular (say C^1) vector field on arbitrary manifolds. Also it is well-known that, if the orbits of a regular vector field are contained on a compact set, then the maximal interval of definition for the corresponding solutions is, indeed, $\mathbb{R}$. In other words the flow generated by this vector field is complete. In particular every regular vector field defined on a compact manifold (without boundary) is complete.

Let us now return to our polynomial vector field $X = (P, Q)$. What precedes implies that the image of Φ decomposes $\mathbb{R}^2$ into a set of curves (orbits of X) along with the singular points of X. To develop this remark further, let us recall the definition of (regular, real) foliations.

Definition 1.1 *Consider a manifold M of real dimension n. A foliation $\mathcal{F}$ of class C^r and of dimension k ($1 \leq k < n$) on M consists of a distinguished coordinate covering $\{U_i, \psi_i\}$, $i \in I$, of M satisfying the conditions below:*

1. *If $i \in I$, then $\psi_i(U_i) = U_i^1 \times U_i^2 \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, where U_i^1, U_i^2 are open discs of $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ respectively.*

2. *If $i, j \in I$ and $U_i \cap U_j \neq \emptyset$, then the change of coordinates $\psi_i \circ \psi_j^{-1} : \psi_j(U_i \cap U_j) \rightarrow \psi_i(U_i \cap U_j)$ has the form $\psi_i \circ \psi_j^{-1}(x, y) = (h_1(x, y), h_2(y))$ where $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{n-k}$.*

Naturally a distinguished coordinate covering for a foliation $\mathcal{F}$ can automatically be enlarged to a maximal *foliated atlas*. A distinguished coordinate $\psi_i : U_i \rightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$ is sometimes called *a foliated chart*, *a foliated coordinate* or even *a trivializing coordinate for $\mathcal{F}$*. Given a foliated chart ψ as above, a set of the form $\psi_i^{-1}(U_i^1 \times \mathrm{cte})$ is called a *plaque*. A *plaque chain* is a sequence of plaques $\alpha_1, \ldots, \alpha_l$ such that $\alpha_i \cap \alpha_{i+1} \neq \emptyset$ for every $i \in \{1, \ldots, l-1\}$. We then introduce an equivalence relation

between points of M by stating that $p \in M$ is equivalent to $q \in M$ if there is a plaque chain $\alpha_1, \ldots, \alpha_l$ such that $p \in \alpha_1$ and $q \in \alpha_l$. The classes of equivalence of this relation are said to be the *leaves* of $\mathcal{F}$.

Remark 1.2 If M is a complex manifold and the changes of coordinates for a foliated atlas are, in fact, holomorphic diffeomorphisms then we have a holomorphic foliation.

With this terminology, we return to the vector field X. A standard fact about Ordinary Differential Equations is the so-called Flow Box Theorem. It states the existence of a diffeomorphism R defined on a neighborhood V of a non-singular point of X, such that $R_* X = e_1$, where $(e_1, \ldots, e_n)$ denotes the canonical basis of $\mathbb{R}^n$. Once again, this result holds for C^r vector fields ($r \geq 1$) and the resulting diffeomorphism R has the same regularity C^r of the vector field.

In particular, away from the singular points of X, the Flow Box Theorem implies that solutions of ODEs are the leaves of a foliation of dimension 1. In this sense, we say that the image of Φ defines a *singular foliation* on $\mathbb{R}^2$. We shall make this definition more formal in the sequel.

1.2 Complex Ordinary Differential Equations

We now wish to extend these notions to the complex case. The main idea is to identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by considering a *complex structure* on $\mathbb{R}^{2n}$. To do this, we must define an automorphism of $\mathbb{R}^{2n}$ that plays the role of the multiplication by $\sqrt{-1}$ in a complex vector space. More precisely, a *complex structure* on $\mathbb{R}^{2n}$ consists of an automorphism $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfying $J^2 = -Id$. In the sequel we are going to use the "standard" complexification, where J is given by $J(x_1, y_1, \ldots, x_n, y_n) = (-y_1, x_1, \ldots, -y_n, x_n)$.

Now, let $X : \mathbb{C}^2 \to \mathbb{C}^2$ given by $X(z_1, z_2) = (P(z_1, z_2), Q(z_1, z_2))$ be a polynomial vector field. The complex ODE associated to this field is

$$\begin{cases} \frac{d}{dT} z_1(T) = P(z_1(T), z_2(T)) \\ \\ \frac{d}{dT} z_2(T) = Q(z_1(T), z_2(T)) \, , \end{cases} \tag{1.2}$$

where the time parameter T is complex. Once again, X being a holomorphic vector field, the complex version of the Theorem of Existence and Uniqueness for regular ODEs guarantees that given $T_0 \in \mathbb{C}$ and $(a, b) \in \mathbb{C}^2$ there exists a unique holomorphic solution $\phi(T) = (\phi_1(T), \phi_2(T))$ of (1.2) defined on a neighborhood B of T_0, and satisfying:

$$\phi(T_0) = (\phi_1(T_0), \phi_2(T_0)) = (a, b) \, .$$

The next step would be to try to glue together these local solutions so as to obtain a "maximal domain of definition". However we notice that, in general, this is not possible since the time parameter T is in $\mathbb{C}$. Indeed, the problem is that,

as we try to glue together the neighborhoods, their union V *may not* be simply connected (see figure below). Consequently, the solution $\phi(T)$ may not be well-defined on all of V, i.e, it may be multi-valued. This is an important difference between real and complex ODEs. This phenomenon is illustrated by Figure (1.2), since the intersection of V_1 and V_2 is not connected, solutions defined on V_1 and V_2 cannot, in general, be "adjusted" to coincide in both connected components.

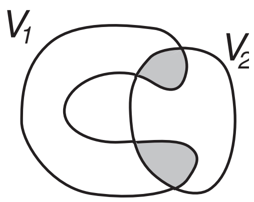

Figure 1.1: $V = V_1 \cup V_2$ is not simply connected.

Let us now introduce a more geometric point of view for these topics. Using the above mentioned identification of $\mathbb{C}$ with $\mathbb{R}^2$ (and $\mathbb{C}^2$ with $\mathbb{R}^4$), a solution of (1.2) starting at $(a, b) = (a_1 + ia_2, b_1 + ib_2)$ may be regarded locally as a "piece" of real 2-dimensional surface L_0 in $\mathbb{R}^4$ passing through the point (a_1, a_2, b_1, b_2). In addition, at (a_1, a_2, b_1, b_2), L_0 is tangent to the vector space spanned by the vectors

$$D_{(0,0)}(\phi_1, \phi_2).\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad D_{(0,0)}(\phi_1, \phi_2).\begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{1.3}$$

where $\phi = (\phi_1, \phi_2)$ is regarded as a real map from $B \subseteq \mathbb{R}^2$ to $\mathbb{R}^4$, and ϕ_i $(i = 1, 2)$ satisfies the Cauchy-Riemann equations. Notice that $T_{(a,b)}L_0$ is invariant under the automorphism J, due to the Cauchy-Riemann equations, so that $T_{(a,b)}L_0$ is indeed a *complex line*, i.e the image of a one-dimensional subspace (over $\mathbb{C}$) of $\mathbb{C}^2 \simeq \mathbb{R}^4$ under the preceding identification.

As the initial conditions (a, b) vary, from (1.3) we obtain a distribution of 2-real dimensional real planes (or complex lines) that can be integrated in the sense of Frobenius to yield 2-dimensional surfaces (or complex curves). In particular, away from its singular set, the vector field defines a foliation of real dimension equal to 2. Furthermore this foliation is holomorphic as it follows from the complex version of the Flow Box Theorem.

The leaves of the foliation in question inherit a natural structure of Riemann surfaces. Actually an atlas for this structure is provided by the local solutions ϕ of (1.2). More precisely, ϕ is a holomorphic diffeomorphism from $B \subseteq \mathbb{C}$ to its image in the leaf (Riemann Surface) L_0.

Summarizing what precedes, a holomorphic vector field on $\mathbb{C}^2$ immediately yields a holomorphic foliation on $\mathbb{C}^2$ away from its singular set. Again we also say that the vector field defines a singular foliation on $\mathbb{C}^2$ which is said to be its associated foliation (or underlying foliation). Conversely, given a (singular) holomorphic foliation $\mathcal{F}$, in order to obtain the vector field whose non-constant orbits are the leaves of

$\mathcal{F}$, we need an extra data. More precisely, we must associate a complex number (or vector in $\mathbb{R}^2$) to the tangent space of each leaf, so as to recover the parametrization of the leaves of $\mathcal{F}$, which in the above situation was given by ϕ. This complex number will be playing the role of the "speed" of the flow of X. It allows us to recover the local parametrizations ϕ for the leaves of the foliation which are given as local solutions of (1.2).

Clearly the notion of singular holomorphic foliation is a convenient geometric way to think of a complex ODE (equivalently a holomorphic vector field). Nonetheless the above remark shows that it does not capture all the information contained in a vector field. As already mentioned, the local solutions ϕ in general cannot be glued together and this makes the problem of extending them, as far as possible, more subtle. We shall return to this problem later.

1.3 Basic Definitions and Examples

After the relatively informal discussion of the previous section, we shall now begin to provide precise definitions and detailed statements.

Definition 1.2 *A complex manifold M^n of dimension n is a differential manifold equipped with an atlas $\{U_i, \psi_i\}$ such that:*

$$\psi_j \circ \psi_i^{-1} : \psi_i(U_i \cap U_j) \to \psi_j((U_i \cap U_j)$$

is holomorphic whenever $U_i \cap U_j \neq \varnothing$, and $\psi_i : U_i \to V_i \subseteq \mathbb{C}^n$, $\psi_j : U_j \to V_j \subseteq \mathbb{C}^n$.

By virtue of the Cauchy-Riemann equations, the Jacobian determinant of a holomorphic diffeomorphism is always positive. It then follows that every complex manifold is orientable.

The Cauchy-Riemann equations also imply that a map $F : U \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is holomorphic if and only if $DF(Jv) = J(DFv)$, for $v \in U$. So that vectors in $\mathbb{R}^{2n}$ invariant under J are sent by DF to vectors that are invariant under J as well. In other words, if F is holomorphic then DF preserves n-dimensional complex planes.

Next, we shall give a working definition of singular holomorphic foliation on a complex manifold which is going to be used throughout these notes.

Definition 1.3 *A singular holomorphic foliation $\mathcal{F}$ defined on a complex manifold M^n consists of the following data:*

1. *There exists an atlas $\{U_i, \psi_i\}$ compatible with the complex structure on M^n, where $\psi_i : U_i \to V_i \subseteq \mathbb{C}^n$.*

2. *There exist holomorphic vector fields X_i defined on each V_i, given by $P_{1,V_i}\frac{\partial}{\partial z_1} + \ldots + P_{n,V_i}\frac{\partial}{\partial z_n}$.*

3. If $U_i \cap U_j \neq \varnothing$ then there exist functions $h_{ij} : \psi_i(U_i \cap U_j) \to \mathbb{C}$ such that:

$$(\psi_j \circ \psi_i^{-1})_* X_i(\psi_i(U_i \cap U_j)) = h_{ij}(z_1, \ldots, z_n) X_j(\psi_j(U_i \cap U_j)) \,.$$

When M is a complex surface (i.e. M has complex dimension 2), we have initially thought of singular foliation as being a regular foliation defined away from finitely many points (the corresponding singularities). In this regard the definition above seems to be more restrictive. Yet this is not the case.

If the functions h_{ij} are actually all constant and equal to 1, then we have, indeed, a holomorphic vector field on M.

Definition 1.4 *A holomorphic vector field X defined on a manifold M^n is such that, given an atlas of M^n, $\{U_i, \psi_i\}$, the following equation is satisfied:*

$$(\psi_j \circ \psi_i^{-1})_* X_i(\psi_i(U_i \cap U_j)) = X_j(\psi_j(U_i \cap U_j)) \,,$$

where $\psi_i : U_i \to V_i \subset \mathbb{C}^n$ and X_i are as in Definition 1.3.

In the sequel we present a list of vector field and foliations of varied nature. Our main purpose is to convince the reader of the richness and importance of the subject. The first elementary examples will also give us a hint that the condition required to define a vector field on a complex manifold is much stronger than the conditions that allow us to define a singular holomorphic foliation.

Example 1: Complex Tori

Let Λ be a lattice on $\mathbb{C}^n$. The n-dimensional complex torus is the quotient space $\mathbb{C}^n/\Lambda$. Notice that a constant vector field Y on $\mathbb{C}^n$ induces a holomorphic vector field on the torus. In fact, constant vector fields are obviously preserved by the translations of $\mathbb{C}^n$ associated to the elements of Λ. Thus Y descends as a holomorphic vector field to the torus given by the quotient $\mathbb{C}^n/\Lambda$.

Example 2: Hopf Surfaces

Consider λ_1, λ_2 in $\mathbb{C}^*$ such that $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Let $\sigma(z_1, z_2) = (\lambda_1 z_1, \lambda_2 z_2)$. The Hopf surface M associated to σ is the quotient $(\mathbb{C}^2 \setminus \{(0,0)\})/\sigma$. It is immediate to check that M is, in fact, a complex 2-dimensional manifold.

Let $X(z_1, z_2) = P(z_1, z_2)\frac{\partial}{\partial z_1} + Q(z_1, z_2)\frac{\partial}{\partial z_2}$ be a polynomial vector field on $\mathbb{C}^2 \setminus \{(0,0)\}$ such that

$$X(\sigma(z_1, z_2)) = \alpha P(z_1, z_2)\frac{\partial}{\partial z_1} + \beta P(z_1, z_2)\frac{\partial}{\partial z_2} \,,$$

where $\alpha/\lambda_1 = \beta/\lambda_2$. Then X defines a singular holomorphic foliation on $(\mathbb{C}^2 \setminus \{0,0\})/\sigma$. The reader will notice, however, that in order to obtain a holomorphic vector field in the Hopf surface, one should have the above ratio equal to 1. This

indicates that in Hopf surfaces, there are many more holomorphic foliations than holomorphic vector fields.

The following concrete example illustrates this. Let $\lambda_1 = e^{-2}$ and $\lambda_2 = e^{-4}$. Consider the polynomial vector field given by $X = P(z_1, z_2)\, \partial/\partial z_1 + Q(z_1, z_2)\, \partial/\partial z_2$ where:

$$\begin{aligned} P(z_1, z_2) &= z_1^3 + z_1 z_2\,, \\ Q(z_1, z_2) &= z_2^2 + 2z_1^2 z_2\,. \end{aligned}$$

Notice that

$$X(\sigma(z_1, x_2)) = e^{-6} P(z_1, z_2) \frac{\partial}{\partial z_1} + e^{-8} Q(z_1, z_2) \frac{\partial}{\partial z_2}\,.$$

On the other hand,

$$D\sigma.X(z_1, z_2) = e^{-2} P(z_1, z_2) \frac{\partial}{\partial z_1} + e^{-4} Q(z_1, z_2) \frac{\partial}{\partial z_2}\,,$$

so that $D\sigma.X(z_1, z_2) = e^4 X(\sigma(z_1, z_2))$. In view of the above discussion, the vector field X induces a holomorphic foliation on M but not a holomorphic vector field.

Example 3: Complex Projective Plane (Space)

Foliations on complex projective spaces constitute the main source of examples, in the sense that they are easy to describe and, in addition, they usually already encode the essential difficulties of more general cases. For this reason, they are going to be treated here with a significative amount of details. Let us begin by considering the following relation of equivalence:

$$z \sim z' \;\Leftrightarrow\; \exists\, \lambda \in \mathbb{C}^* \qquad ; \qquad z = \lambda z'; \qquad z, z' \in \mathbb{C}^3 \setminus \{(0,0,0)\}\,.$$

The equivalence classes $(\mathbb{C}^3 \setminus \{(0,0,0)\})/\sim$ form the 2-dimensional complex projective space, denoted by $\mathbb{CP}(2)$.

Two points $(a, b, c), (a', b', c') \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ define the same point in $\mathbb{CP}(2)$ if and only if

$$a/a' = b/b' = c/c'\,,$$

so that the projection $\pi : \mathbb{C}^3 \setminus \{(0,0,0)\} \to \mathbb{CP}(2)$ is determined by the ratios between the coordinates of (a, b, c). Traditionally $\pi(a, b, c)$ is denoted by $(a : b : c)$ and a, b, c are called homogeneous coordinates for $(a : b : c)$.

Let us consider the following open sets that cover $\mathbb{CP}(2)$:

$$\begin{aligned} U_a &= \{(a : b : c) \in \mathbb{CP}(2) \;\; ; \;\; a \neq 0\}\,; \\ U_b &= \{(a : b : c) \in \mathbb{CP}(2) \;\; ; \;\; b \neq 0\}\,; \\ U_c &= \{(a : b : c) \in \mathbb{CP}(2) \;\; ; \;\; c \neq 0\}\,. \end{aligned}$$

Along with these open sets, we have the following coordinate charts:

$$\begin{aligned} \varphi_a : U_a &\to \mathbb{C}^2 \\ (a : b : c) &\mapsto (b/a, c/a) = (x, y)\,; \end{aligned}$$

$$\begin{array}{rcl} \varphi_b : U_b & \to & \mathbb{C}^2 \\ (a:b:c) & \mapsto & (a/b, c/b) = (u,v)\,; \end{array}$$

$$\begin{array}{rcl} \varphi_c : U_c & \to & \mathbb{C}^2 \\ (a:b:c) & \mapsto & (a/c, b/c) = (z,w)\,. \end{array}$$

Now let $L_\infty = \mathbb{CP}\,(2) \setminus E_a = \{(0:b:c) \in \mathbb{CP}\,(2); \quad (b,c) \in \mathbb{C}^2\}$. A direct inspection using the coordinate charts introduced above shows that L_∞ is isomorphic to $\mathbb{CP}\,(1)$. Thus $\mathbb{CP}\,(2) = \mathbb{C}^2 \cup \mathbb{CP}\,(1)$ i.e. $\mathbb{CP}\,(2)$ may be regarded as $\mathbb{C}^2$ being added to the Riemann sphere. In this sense $\mathbb{CP}\,(2)$ is a compactification of $\mathbb{C}^2$. Moreover in the affine coordinates (x,y), L_∞ corresponds to "infinity" and L_∞ is linearly embedded in $\mathbb{CP}\,(2)$. For these reasons it is called the *line at infinity.* Finally, we note that this construction applies to any affine coordinates in $\mathbb{CP}\,(2)$, that is, any affine $\mathbb{C}^2 \subset \mathbb{CP}\,(2)$ gives rise to a "line at infinity".

We will construct singular holomorphic foliation on $\mathbb{CP}\,(2)$ in a similar way to what was done in the case of Hopf surfaces.

Let us consider a homogeneous polynomial vector field $X = P\frac{\partial}{\partial z_1} + Q\frac{\partial}{\partial z_2} + R\frac{\partial}{\partial z_3}$ on $\mathbb{C}^3 \setminus \{(0,0,0)\}$ of degree d. Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^3 \setminus \{0,0,0\}$ given by the homotheties $\sigma(z_1,z_2,z_3) = (\lambda z_1, \lambda z_2, \lambda z_3)$, $\lambda \in \mathbb{C}^*$. As already pointed out, the quotient of this action is precisely $\mathbb{CP}\,(2)$. On the other hand, note that

$$X(\sigma(z_1,z_2,z_3)) = \lambda^d P(z_1,z_2,z_3)\frac{\partial}{\partial z_1} + \lambda^d Q(z_1,z_2,z_3)\frac{\partial}{\partial z_2} + \lambda^d R(z_1,z_2,z_3)\frac{\partial}{\partial z_3}\,.$$

Besides,

$$D\sigma.X(z_1,z_2,z_3) = \lambda P(z_1,z_2,z_3)\frac{\partial}{\partial z_1} + \lambda Q(z_1,z_2,z_3)\frac{\partial}{\partial z_2} + \lambda R(z_1,z_2,z_3)\frac{\partial}{\partial z_3}\,.$$

Therefore $D\sigma.X = \lambda^{d-1}X$ so that "the directions" associated to X are invariant under homotheties. Hence X indeed defines a singular foliation on $\mathbb{CP}\,(2)$.

Another equivalent way to define a singular holomorphic foliation in $\mathbb{CP}\,(2)$ is to consider a polynomial vector field in $\mathbb{C}^2$ and see that it can be extended to an holomorphic foliation on all of $\mathbb{CP}\,(2)$. Details are provided below.

In fact, it can be shown that every holomorphic foliation in $\mathbb{CP}(2)$ is induced by a polynomial vector field in $\mathbb{C}^2$.

Let $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ be a polynomial vector field in the affine coordinates (x,y). Naturally, it induces a rational vector field Y (resp.Z) defined on the affine coordinates (u,v) (resp. (z,w)). Then we only need to multiply the vector fields Y,Z by their denominators so as to have holomorphic ones (indeed polynomial ones).

For example, Y is given by:

$$\begin{array}{rcl} Y(u,v) & = & (\varphi_a \circ \varphi_b)^*(X(x,y)) \\ & = & D(\varphi_a \circ \varphi_b)^{-1}.X(\varphi_a \circ \varphi_b(u,v)) \\ & = & \begin{pmatrix} -u^2 & 0 \\ -uv & u \end{pmatrix} \begin{pmatrix} P(1/u, v/u) \\ Q(1/u, v/u) \end{pmatrix}. \end{array}$$

As already mentioned, this vector field is not holomorphic in the domain of the coordinates (u, v) since it has poles over $\{u = 0\}$. However, multiplying Y by u^d, the new vector field $u^d Y$ is obviously holomorphic (and with isolated singularities) in the domain of (u, v). Now changing from the coordinate system (u, v) to (x, y) one obtains

$$\begin{aligned} D(\varphi_a \circ \varphi_b).(u^d Y) &= u^d D(\varphi_a \circ \varphi_b).Y(u, v) \\ &= u^d D(\varphi_a \circ \varphi_b).D(\varphi_a \circ \varphi_b)^{-1}.X(x, y) \\ &= u^d X(x, y)\,. \end{aligned}$$

Repeating this procedure with the other coordinate charts, we obtain a holomorphic foliation on $\mathbb{CP}(2)$. Finally note also that the original polynomial vector field on $\mathbb{C}^2$ does not induce a holomorphic vector field on $\mathbb{CP}(2)$. Instead it induces a *meromorphic* vector field whose poles are contained in the corresponding line at infinity.

This has an obvious generalization to higher dimensional complex projective spaces which is left to the reader.

We have constructed singular holomorphic foliations on $\mathbb{CP}(2)$ following two *a priori* different methods:

- By means of a homogeneous polynomial vector field on $\mathbb{C}^3$.
- By means of a polynomial vector field on $\mathbb{C}^2$.

It is easy to check that both constructions are equivalent in the sense that they produce the same set of foliations. This verification is implicitly carried out in the proof of Lemma (1.2). It is however much harder to show that these constructions give rise to *all singular holomorphic foliations on* $\mathbb{CP}(2)$. This is the contents of Theorem (1.1) below.

Theorem 1.1 *Let $\mathcal{F}$ be a singular holomorphic foliation on $\mathbb{CP}(2)$. Then there is $\mathcal{F}$ is induced by a homogeneous polynomial vector field X on $\mathbb{C}^3$ which, in addition, has singular set of codimension at least* 2.

In view of Theorem (1.1), it is natural to try to define a notion of *degree* for a foliation on $\mathbb{CP}(2)$. At first, one might feel tempted to define it as the degree of a polynomial vector field representing the foliation in affine coordinates. However, as the reader can easily verify, this degree may, in fact, vary depending on the affine coordinates chosen.

The following lemma will motivate the correct definition of the degree for a foliation $\mathcal{F}$ on $\mathbb{CP}(2)$ induced by a polynomial vector field $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ on $\mathbb{C}^2$. Let $P = \sum_{i=0}^{d} P_i(x, y)$ and $Q = \sum_{i=0}^{d} Q_i(x, y)$ where P_i, Q_i are homogeneous polynomials of degree i.

Lemma 1.1 *The "line at infinity", L_∞, of $\mathbb{CP}(2)$ is invariant under the foliation $\mathcal{F}$, induced by X as above, if and only if the top-degree homogeneous component $P_d\frac{\partial}{\partial x} + Q_d\frac{\partial}{\partial y}$ is not of the form $h(x, y)(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$, for some polynomial h of degree $d - 1$.*

Proof. To understand the behavior of $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ near infinity in the coordinate system (x, y), we use the following change of coordinates: $u = \frac{1}{x}$, $v = \frac{y}{x}$. So that this vector field in the coordinate chart (u, v) is given by:

$$\begin{aligned} X(u,v) &= \begin{pmatrix} -u^2 & 0 \\ -uv & u \end{pmatrix} \begin{pmatrix} P(1/u, v/u) \\ Q(1/u, v/u) \end{pmatrix} \\ &= -u^2(P(1/u, v/u))\frac{\partial}{\partial u} + u(Q(1/u, v/u) - vP(1/u, v/u))\frac{\partial}{\partial v} \end{aligned}$$

and now we only need to analyze the corresponding foliation on a neighborhood of $\{u = 0\}$.

Let us denote $\sum_{i=0}^{d-1} P_i$ by $\widetilde{P}(1/u, v/u)$, and $\sum_{i=0}^{d-1} Q_i$ by $\widetilde{Q}(1/u, v/u)$. Notice that $u^d\widetilde{P}(1/u, v/u) = uh_1(u, v)$, $u^d\widetilde{Q}(1/u, v/u) = uh_2(u, v)$ for appropriate polynomials h_1 and h_2 of degree $d - 1$. Also, $u^dP_d(1/u, v/u) = P_d(1, v)$, naturally there is an analogous expression for Q_d. Multiplying the vector field $X(u, v)$ by u^{d-1} we obtain a holomorphic vector field in the (u, v)-coordinate which is given by

$$Y(u,v) = -u(P_d(1,v) + uh_1(u,v))\frac{\partial}{\partial u} + (Q_d(1,v) - vP_d(1,v) + ug(u,v))\frac{\partial}{\partial v},$$

where $g(u, v) = h_2(u, v) - vh_1(u, v)$.

Now, if $Q_d(1, v) - vP_d(1, v) \equiv 0$, the components of $Y(u, v)$ are both divisible for u. By eliminating this common factor, it becomes clear that the line at infinity is not preserved by the foliation. Conversely, if $Q_d(1, v) - vP_d(1, v)$ is not identically zero, L_∞ is preserved, since the component $\partial/\partial u$ of Y vanishes identically over $L_\infty \simeq \{u = 0\}$.

Finally it is clear that $Q_d(1, v) - vP_d(1, v)$ vanishes identically if and only if the top-degree homogeneous component is radial. ∎

Definition 1.5 *The degree of a foliation $\mathcal{F}$ on $\mathbb{CP}(2)$ given by the compactification of the polynomial vector field $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ of degree d and having only isolated zeros is equal to:*

1. *$d - 1$, if there exists a polynomial $h(x, y)$ of degree $d - 1$ such that $P_d\frac{\partial}{\partial x} + Q_d\frac{\partial}{\partial y} = h(x, y)(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$. In other words, $d-1$ if the top-degree homogeneous component of X is a multiple of the radial vector field.*

2. *d, otherwise.*

It should be verified that this is indeed well-defined.

Here we shall give a more geometric interpretation of the degree of a foliation as defined above. In fact, the contents of this lemma can be used as an equivalent definition of degree.

Lemma 1.2 *Let $\mathcal{F}$ be a singular holomorphic foliation on $\mathbb{CP}(2)$ of degree d. Then the following holds:*

1. *There is a homogeneous polynomial vector field Z of degree d on $\mathbb{C}^3$, with singular set of codimension at least* 2*, which induces $\mathcal{F}$ by radial projection of its orbits;*

2. *The number of tangencies of $\mathcal{F}$ with a generic projective line is d.*

Proof. Let us first show that the projection of the foliation associated to a homogeneous polynomial vector field of degree d on $\mathbb{C}^3$ is, indeed, a foliation $\mathcal{F}$ on $\mathbb{CP}(2)$ having degree d. Let $Z = \sum_{i=0}^{2} H_i(z_0, z_1, z_2)\frac{\partial}{\partial z_i}$, where (z_0, z_1, z_2) stands for the coordinates of $\mathbb{C}^3$. In the chart $(x, y) = (z_1/z_0, z_2/z_0)$, the vector field Z becomes

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y} \ , \quad \text{where}$$

$$\begin{aligned} P(x,y) &= H_1(1,x,y) - xH_0(1,x,y) \\ Q(x,y) &= H_2(1,x,y) - yH_0(1,x,y)\,. \end{aligned}$$

If $H_0(1, x, y)$ has degree d (i.e, H_0 is not divisible by z_0), then X has degree $d+1$. Furthermore, the top-degree component of X is given by $-H_0^d(1, x, y)(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$, where H_0^d stands for the homogeneous component of degree d of $H_0(1, x, y)$. So, according to Definition (1.5), $\mathcal{F}$ has degree d.

If $H_0(1, x, y)$ has degree less than d (i.e, H_0 is divisible by z_0), then at least one between $H_1(1, x, y)$ and $H_2(1, x, y)$ must have degree d, otherwise all of the three polynomials would be divisible by z_0. This would mean that the singular set of Z has codimension 1, which contradicts our assumption. Therefore, X has degree d, and once again Definition (1.5) implies that $\mathcal{F}$ has degree d. The converse is analogous and establishes the first part of the statement.

Let us now consider the tangencies between $\mathcal{F}$ and a generic line in $\mathbb{CP}(2)$. Modulo performing a projective change of coordinate, we may suppose that the tangencies with a generic projective line $y = \lambda x$ are all contained in the main affine chart of $\mathbb{CP}(2)$ so that they are given by:

$$\lambda P(x, \lambda x) = Q(x, \lambda x)\,,$$

that is, by the zeros of the polynomial $\lambda P(x, \lambda x) - Q(x, \lambda x)$. Hence the number of tangencies (counted with multiplicity) is the degree of $\lambda P(x, \lambda x) - Q(x, \lambda x)$. However, if d is the degree of the foliation, then either $P_d(x, y)\frac{\partial}{\partial x} + Q_d(x, y)\frac{\partial}{\partial y}$ is radial (and consequently X has degree $d + 1$) or it is not (and X has degree d). The first case is equivalent to have $\lambda P_d(x, \lambda x) - Q_d(x, \lambda x) = 0$, which means that $\lambda P(x, \lambda x) - Q(x, \lambda x)$ has degree d. The other case only happens when the top-degree component of $\lambda P(x, \lambda x) - Q(x, \lambda x)$ is not zero, implying that the degree of this polynomial is d. The converse is again analogous. ■

According to Theorem (1.1) and to Lemma (1.2), the space $\mathrm{Fol}\,(\mathbb{CP}(2), d)$ consisting of singular holomorphic foliation of degree d is naturally contained in the

space of homogeneous polynomial vector fields of degree d on three variables. Besides two such vector fields having a singular set of codimension at least 2 define the same foliation if and only if they differ by a multiplicative constant. Thus a simple counting of coefficients yields the following corollary:

Corollary 1.1 *The space* $\mathrm{Fol}\,(\mathbb{CP}\,(2), d)$ *is naturally identified with a Zariski-open set of the complex projective space of dimension*

$$(d+1)(d+3) - 1\,.$$

□

It should also be noted that the group of automorphims of $\mathbb{CP}\,(2)$, $\mathrm{PSL}\,(3, \mathbb{C})$, has a natural action on $\mathrm{Fol}\,(\mathbb{CP}\,(2), d)$ through projective changes of coordinates.

Example 4: Foliations on weighted projective spaces

This is a very natural generalization of the previous example that often appears in higher dimensional questions related to resolution of singularities, cf. for example [A]. They are also similar to foliations on Hopf surfaces.

A polynomial P on n variables $(x_1, \ldots, x_n)$ is said to be *quasi-homogeneous* with *weights* $(k_1, \ldots, k_n)$ and degree d if and only if for every $\lambda \in \mathbb{C}^*$ one has

$$P(\lambda^{k_1} z_1, \ldots, \lambda^{k_n} z_n) = \lambda^d P(z_1, \ldots, z_n)\,.$$

For example $P(x, y, z) = xz + y^2$ is not only quadratic (ie homogeneous of degree 2) but also quasi-homogeneous of degree 4 relative to the weights $(1, 2, 3)$.

Chosen a set of weights $(k_1, \ldots, k_n)$, we have a natural action of $\mathbb{C}^*$ on $\mathbb{C}^n \setminus \{(0, \ldots, 0)\}$ given by

$$\lambda.(z_1, \ldots, z_n) = (\lambda^{k_1} z_1, \ldots, \lambda^{k_n} z_n)\,.$$

Consider the quotient space of $\mathbb{C}^n \setminus \{(0, \ldots, 0)\}$ where two points are identified if and only if they belong to the same orbit of $\mathbb{C}^*$. The resulting space is a compact manifold with singularities called a *weighted projective space* whose dimension is obviously equal to $n - 1$. Whether or not it actually has singularities, this type of manifold can be given an algebraic structure since it can be realized as a Zariski-closed set of a complex projective space with sufficiently high dimension. The existence of this imbedding can easily be shown by means of Plücker coordinates.

Alternatively the quotient of this $\mathbb{C}^*$-action can also be realized as the quotient of the projective space of dimension $n - 1$ by some *finite group of automorphism.*

Next a polynomial vector field $P_1 \partial/\partial x_1 + \cdots + P_n \partial/\partial x_n$ is said to be *quasi-homogeneous* with weights $(k_1, \ldots, k_n)$ and degree d if and only if for every $\lambda \in \mathbb{C}^*$ one has

$$\Lambda^* X = \lambda^{d-1} X\,,$$

where Λ stands for the map $(z_1, \ldots, z_n) \mapsto (\lambda^{k_1} z_1, \ldots, \lambda^{k_n} z_n)$. An example of quasi-homogeneous vector field with weights $(1, 2, 3)$ and degree 4 is

$$(xz + y^2)\frac{\partial}{\partial x} + (2zy + 3x^5)\frac{\partial}{\partial y} + (x^3 z - y^3 + 2z^2)\frac{\partial}{\partial z} \, .$$

If X is quasi-homogeneous with weights $(k_1, \ldots, k_n)$ then the definition above implies in particular that X unequivocally defines a complex direction at each point of the projective space with the same weights $(k_1, \ldots, k_n)$. Being of complex dimension 1, these directions can naturally be integrated to form a singular holomorphic foliation. Therefore quasi-homogeneous vector fields give rise to singular holomorphic foliations on weighted projective spaces.

Example 5: Riccati Foliation The classical Riccati equation is given by

$$\frac{dy}{dx} = a(x)y^2 + b(x)y + c(x) \, ,$$

where x, y are complex variables. We are interested in the case where a, b, c are rational functions of x. In this case, if P denotes the least common multiple of the denominators of $a, b, c,$, the preceding equation is equivalent to the vector field

$$X = P(x)\partial/\partial x + (a^*(x)y^2 + b^*(x)y + c^*(x))\partial/\partial y \, ,$$

with a^*, b^*, c^* polynomials. Although it is possible to compactify the associated foliation on $\mathbb{CP}\,(2)$, it is more natural to compactify it in $\mathbb{CP}(1) \times \mathbb{CP}(1)$. Consider the "vertical" projection π_1 onto the first factor. The change o coordinate $(x, y) \mapsto (x, 1/y)$ allows one to check that the vector field X has a holomorphic extension to the "infinite" of the fibers (or to the section at infinity). However X has in general poles on the "fiber over infinity" of affine coordinates $(1/x, y)$. In the initial affine coordinates (x, y), we see that the set $\{P = 0\}$ is consituted by *invariant fibers* and contains all the (affine) singularities of the underlying foliation $\mathcal{F}$. A similar calculation applies to the fiber over infinity. Thus we conclude that the foliation associated to X has the following properties:

1. It admits a *nonzero* number of invariant fibers given in the affine coordinates (x, y) by $\{P = 0\}$. The fiber over infinity may or may not be invariant by this foliation.

2. The union of these invariant fibers contains all the singularities of $\mathcal{F}$.

3. Away from the invariant fibers, $\mathcal{F}$ is transverse to the fibers of π_1.

Since the fibers of π_1 are compact, a simple remark due to Ehresmann guarantees that the restriction of π_1 to the regular leaves of $\mathcal{F}$ (different from the invariant fibers) defines a covering map onto $\mathbb{CP}(1) \setminus \{p_1, \ldots, p_k\}$ where $p_1, \ldots, p_k \in \mathbb{CP}(1)$ are precisely the projection of the fibers invariant under $\mathcal{F}$.

Fix a fiber $\pi_1^{-1}(p)$, with $p \notin \{p_1, \ldots, p_k\}$. Thanks to the remark above, paths contained in $\mathbb{CP}(1) \setminus \{p_1, \ldots, p_k\}$ can be lifted in the leaves of $\mathcal{F}$. Therefore we can consider the *global holonomy* of $\mathcal{F}$ (w.r.t. π_1). This is given by a homomorphism

$$\rho \Pi_1(\mathbb{CP}(1) \setminus \{p_1, \ldots, p_k\}) \longrightarrow \mathrm{Aut}\,(\pi_1^{-1}(p)) \simeq \mathrm{Aut}\,(\mathbb{CP}(1))\,.$$

Since $\mathrm{Aut}\,(\mathbb{CP}(1))$ is isomorphic to $\mathrm{PSL}\,(2, \mathbb{C})$, the homomorphism ρ can also be viewed as taking values in this latter group. In particular the theory of the Riccati equations is naturally connected to the theory of finitely generated subgroups of $\mathrm{PSL}\,(2, \mathbb{C})$. In particular, in the cases where the group is in addition discrete, to the theory of Kleinian groups and consequently, at least to some extent, to hyperbolic geometry in dimension 3. We also note that, conversely, a classical result due to Birkhoff states that every finitely generated subgroup of $\mathrm{PSL}\,(2, \mathbb{C})$ can be relized as the monodromy group of some Riccati equation. As a matter of fact, this correspondence is not unique: there are several Riccati equations possessing the same monodromy group. This raises the question of trying to find the “simplest” Riccati equation realizing a given monodromy group. As far as we know, this question has not been addressed to in the literature.

Example 6: Linear Equations

Linear equations are among the most classical topics in complex analysis. They generalize Riccati equations as well as many other equations such as Gauss hypergeometric equation, Fuchsian equations and so on. Furthermore they play a significative role in the theory of vector bundles over Riemann surfaces.

In a classical setting we consider a meromorphic function from $\mathbb{C}$ with values on $\mathrm{GL}\,(n, \mathbb{C})$, for some $n \geq 2$. These are simply a family of invertible matrices of rank n whose coefficients are meromorphic functions defined from $\mathbb{C}$ to $\mathbb{C}$. The notion of invertible matrices of course refer to those points away from the poles of these coefficients.

Example 7: Jouanolou Foliation

This is a very special example of foliation on the complex projective plane. Fixed $n \in \mathbb{N}^*$, the *Jouanolou foliation J_n of degree n* is the foliation induced on $\mathbb{CP}\,(2)$ by the homogeneous 1-form

$$\Omega = (yx^n - z^{n+1})\,dx \;+\; (zy^n - x^{n+1})\,dy \;+\; (xz^n - y^{n+1})\,dz\,.$$

The reader will notice that the kernel of Ω always contains the radial direction so that it naturally induces a line field, and hence a foliation, on $\mathbb{CP}\,(2)$. In terms of homogeneous vector field, the foliation J_n is given by

$$y^n \frac{\partial}{\partial x} + z^n \frac{\partial}{\partial y} + x^n \frac{\partial}{\partial z}\,.$$

In standard affine coordinates, $\{z = 1\}$, the foliation is induced by the vector field

$$(y^n - x^{n+1}) \frac{\partial}{\partial x} + (1 - yx^n) \frac{\partial}{\partial y}$$

The main result of Jouanolou states that J_n leaves *no algebraic curve* invariant. Foliations leaving algebraic curves invariant will be discussed in detail in Chapter 3 and the foliations J_n will play a significative role in the discussion. For the time being, let us only mention some elementary properties of J_n. The proofs of the assertions below amount to direct inspection in the spirit of the preceding example, they are therefore left to the reader.

We begin$\{x = y = z\}$ by noticing that J_n is left invariant by a nontrivial group of automorphisms of $\mathbb{CP}(2)$. To identify this group, let ζ be a primitive N^{th}-root of the unit with $N = n^2 + n + 1$. The cyclic group H generated by the automorphism $T(x, y, z) = (\zeta z, \zeta^{-n} y, z)$ preserves J_n as well as it does the group K generated by the cyclic permutation $S(x, y, z) = (y, z, x)$. The order of H is precisely $n^2 + n + 1$ while the order of K is obviously equal to 3. It can be checked that T and S generate the maximal group of automorphisms of $\mathbb{CP}(2)$ preserving J_n.

It is a remarkable fact that all the foliations J_n, $n \in \mathbb{N}^*$, are obtained by means of a family of quadratic foliations on $\mathbb{CP}(2)$. This can be done as follows. Consider that map $\Upsilon_n : \mathbb{CP}(2) \to \mathbb{CP}(2)$ given in homogeneous coordinates by $\Upsilon_n(x : y : z) = (y^{n+1}z : z^{n+1}x : x^{n+1}y)$. This induces a ramified covering of $\mathbb{CP}(2)$ with degree $N = n^2 + n + 1$. It can be checked that J_n is the pull-back $\Upsilon_n \mathcal{Q}_n$ where $\mathcal{Q}_n$ is the quadratic foliation on $\mathbb{CP}(2)$ induced by the homogeneous vector field

$$X_n = x(x - ny)\frac{\partial}{\partial x} + y(y - nz)\frac{\partial}{\partial y} + z(z - nx)\frac{\partial}{\partial z}\ .$$

The reader will immediately check that $\mathcal{Q}_n$ has exactly 7 singularities. These correspond to the radial lines invariant by X_n, namely the coordinates axes, $\{x = y = z\}$, $t.(0, n + 1, 1)$, $t.(n + 1, 1, 0)$ and $t.(1, 0, n + 1)$ where $t \in \mathbb{C}$. $\mathcal{Q}_n$ also leaves 3 projective lines l_1, l_2, l_3 invariant, namely those induced by projection of the coordinates 2-planes. It is easy to see that the lines l_1, l_2, l_3 form a "triangle" containing the singularities of $\mathcal{Q}_n$ except for the singularity $\{x = y = z\}$. This "triangle" is the "maximal" algebraic curve invariant by $\mathcal{Q}_n$.

Naturally X_n is still invariant by the permutation S, and hence by the group K. It is therefore possible to further "simplify" $\mathcal{Q}_n$ by exploting these symmetries. This would lead us to a foliation of higher degree leaving a single irreducible algebraic curve invariant. Further information on Jouanolou foliations can be found in Chapter 3 along with some specific references.

Example 9: Ramanujan Differential Equation

This is a system of differential equations introduced by Ramanujan that plays a significative role in the part of Number Theory dealing with transcendent numbers. The system closed related to the following vector field defined on $\mathbb{C}^3$.

$$X = \frac{1}{12}(x^2 - y)\frac{\partial}{\partial x} + \frac{1}{3}(xy - z)\frac{\partial}{\partial y} + \frac{1}{2}(xz - y^2)\frac{\partial}{\partial z}\,.$$

This vector field is, indeed, a particular example of class of differential equations known as Halphen vector fields. These were detailed studied by A. Guillot in [Gu].

In later chapters, we shall have occasion to talk more about Halphen vector fields. For the time being, we would like to indicate very briefly the connection between the above vector field and the theory of transcendent numbers, for further information we refer the reader to [Ma-Ro]. This begins with the so-called Ramanujan P, Q, R functions that can be defined by explicit formulas as follows.

$$
\begin{aligned}
P(t) &= 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)t^n \, , \\
Q(t) &= 1 + 240\sigma_{n=1}^{\infty}\sigma_3(n)t^n \, , \\
R(t) &= 1 - 504\sigma\sum_{n=1}^{\infty} \sigma_5(n)t^n \, ,
\end{aligned}
$$

where $t \in \mathbb{C}$ and $\sigma_k(n) = \sum_{d|n} d^k$, ie $\sigma_k(n)$ is the sum of the $k^{\rm th}$-powers of the positive integers dividing n. Since these formulas may look especially weird, let us mention that they can be obtained from rather natural procedures with automorphic functions. For our present purpose however the above definition will suffice. For example, it makes easy to see that these functions are defined for $\|t\| < 1$. It is less easy to see that the unit circle constitutes an "essential boundary for them" in the sense that the have no holomorphic extension to a neighborhood of any point in the circle. The interest of these functions for number theorists is partially due to the fact that they assume "specially remarkable values" for specific choices of t.

According to Ramanujan, the functions P, Q, R verify the following *nonautonomous* system of differential equations:

$$
\begin{aligned}
tP'(t) &= \frac{1}{12}(P^2 - Q) \, , \\
tQ'(t) &= \frac{1}{3}(PQ - R) \, , \\
tR'(t) &= \frac{1}{2}(PR - Q^2) \, .
\end{aligned}
$$

Although this system is not autonomous, it ensures that the "velocity vector" of the parametrized curve $t \mapsto (P(t), Q(t), R(t))$ is parallel to X at the point $(P(t), Q(t), R(t))$. In other words, this curve parametrizes a leaf of the foliation $\mathcal{F}$ associated to X. Actually, the image of this curve also contains the point $(1, 1, 1)$ that technically speaking does not belong to the leaf in question since $(1, 1, 1)$ is a singular point of X.

If we choose $t_0 \neq 0$ (for example $t_0 =$) and the corresponding point $(P(t_0), Q(t_0), R(t_0))$, we can compare the functions P, Q, R with the actual solutions $(x(t), y(t), z(t))$ of the vector field X satisfying the initial value condition $(x(t_0), y(t_0), z(t_0)) = (P(t_0), Q(t_0), R(t_0))$. This would lead us to the simple (1-dimensional) equations

$x' = tP'$, $y' = tQ'$ and $z' = tR'$. Therefore the solutions are

$$x(t) = tP(t) + (1 - t_0)P(t_0) - \int_{t_0}^{t} P(s)ds\,.$$
$$y(t) = tQ(t) + (1 - t_0)Q(t_0) - \int_{t_0}^{t} Q(s)ds\,.$$
$$z(t) = tR(t) + (1 - t_0)R(t_0) - \int_{t_0}^{t} R(s)ds\,.$$

Naturally the effect of changing the initial data $t_0 \neq 0$ and $(P(t_0), Q(t_0), R(t_0))$ essentially amounts to performing a translation for the functions (x, y, z) and thus is of little importance.

Summarizing the preceding discussion, we conclude that information on the properties of P, Q, R can be extract from the study of the solutions of the vector field X.

Example 10: The Lorenz Attractor

In the course of the past few years, the dynamical system associated to Lorenz attractor has become a paradigmatic example of "chaotic dynamics". The vector field defining the dynamical system was introduced by Lorenz in [L] in connection with a evolution problem for atmospheric conditions. In general, the domain belongs to fluid dynamics and the phenomenon is governed by the KdV equation. Since these equations are notoriously hard to be analysed, Lorenz thought of his vector field as a simplified model to describe this evolution. The vector field in question is simply given by

$$X = 10(y - x)\frac{\partial}{\partial x} + (28x - y - xz)\frac{\partial}{\partial y} + (xy - 8z/3)\frac{\partial}{\partial z}\,.$$

It is therefore a quadratic vector field (obviously X is not homogeneous so that "quadratic" here is a abuse of language) defined on $\mathbb{R}^3$. Despite its innocent appearance, S exhibits a remarkably complicated dynamics. A traditional picture of the plotting of orbits of X looks like figure 1.2.

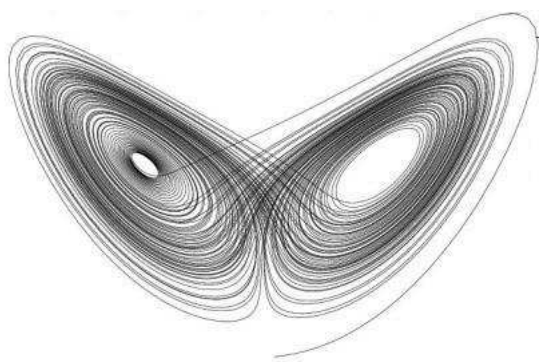

Figure 1.2:

Figure 1.2 seems to indicate the presence of an "attractor" for X. The definition of the term "attractor" may vary depending on the context and we shall give none in this discussion. In any case the guiding picture to bear in mind is that of a "compact invariant set" attracting nearby orbits. In particular, orbits of points that actually belong to the attractor remain contained in it forever (both in "past" and in "future"). Since this attractor is not hyperbolic, it was called "strange". Here the reader may consider that hyperbolicity is the best known mechanism to produce this type of behavior. Besides, it has been established that the attractor, if it existed, was *robust*, ie. it could not be destroyed by performing a small perturbation on X. Curiously enough, the very existence of Lorenz attractor was settled only recently by W. Tucker through rigorous numerics [T]. An interesting survey on the Lorenz attractor containing in particular precise definitions and notions is [V]. For a beautiful geometric discussion of the properties of this attractor, including some wonderful graphics, the reader is referred to [G-L].

Although the interest on Lorenz vector field has primarily to do with its real dynamics, this vector field can equally well be thought of as being a complex polynomial (and thus holomorphic) vector field defined on $\mathbb{C}^3$. It is natural to consider this complexification not only for it may provide new insight in the real dynamics, but also because it is likely to be interesting in itself. Besides, in view of Example 3, it may be useful to consider the holomorphic extension to $\mathbb{CP}(3)$ of the foliation $\mathcal{F}$ associated to X.

Elementary calculations yield some basic facts, such as singularities and existence of invariant curves, about $\mathcal{F}$ viewed as a singular holomorphic foliation on $\mathbb{CP}(3)$. In the affine $\mathbb{C}^3$, the foliation has exactly three singularities corresponding to the points $(0,0,0)$, $(6\sqrt{2}, 6\sqrt{2}, 27)$ and $(-6\sqrt{2}, -6\sqrt{2}, 27)$ which in addition have non-degenerate linear part. Also the axis $\{x = y = 0\}$ is obviously invariant by $\mathcal{F}$. The plane at infinity $\Delta = \mathbb{CP}(3) \setminus \mathbb{C}^3$ is also invariant by $\mathcal{F}$ and contains 3 other singularities of it. In coordinates $x = 1/u$, $y = v/u$ and $z = w/u$, the plane at infinity is given by $\{u = 0\}$ the restriction of $\mathcal{F}$ to this plane coincides with the one induced by the vector field $-28w\partial/\partial v + v\partial/\partial w$. In particular, it has a singularity whose eigenvalues vanish at $u = v = w = 0$. The complement of the coordinates v, w in Δ is a projective line that is, in addition, invariant by $\mathcal{F}$. This line contains the last two singularities of $\mathcal{F}$.

The picture of the existence of the Lorenz attractor and the consequently existence of real orbits lying entirely in some compact set of $\mathbb{R}^3$ raises the question of knowing if a similar phenomenon will happen with the complex leaves of $\mathcal{F}$. This is however not the case. Still the question serves as motivation for us to state and proof Proposition 1.1 below. This proposition will play a significative role in Chapter 4 as well as it does in many aspects of the study of holomorphic foliations on complex projective spaces.

Proposition 1.1 *Let $\mathcal{F}$ be a holomorphic foliation of $\mathbb{CP}(n)$ and consider the foliation $\mathcal{F}^0$ obtained as the restriction of $\mathcal{F}$ to an affine $\mathbb{C}^n \subset \mathbb{CP}(n)$. Then no regular leaf of $\mathcal{F}^0$ can entirely be contained in a compact set $K \subset \mathbb{C}^n$.*

It is to be noted that this proposition cannot be derived as an immediate consequence of the maximum principle since the orbits of $\mathcal{F}^0$ need not be compact. Indeed, in the compact case, we can consider the projection of the compact leaf on a chosen coordinate and argument that the image of this projection must be a single point by virtue of the maximum principle. In fact, there must be a point in the leaf corresponding to a projection of "maximal modulus". It then follows that the projection in question is constant. In the case of open leaves, it is not obvious that the maximum of a projection as above is attained so that we cannot conclude that the projection is constant.

Proof. Suppose for a contradiction that L is a regular leaf of $\mathcal{F}^0$ that is wholly contained in a compact set $K \subset \mathbb{C}^3$. Consider a polynomial vector field X tangent to $\mathcal{F}$. The leaf L of $\mathcal{F}$ can then be considered as an orbit of X. Since X is uniformly bounded in K, there exists $\epsilon > 0$ such that for every $p \in K$ the solution φ of X satisfying the initial condition $\varphi(t) = p$ is defined on a disc of radius r about $t \in \mathbb{C}$. Now we proceed by choosing a point $p_1 \in L \cap K$ and considering the solution φ_1 satisfying $\varphi_1(t_1) = p_1$, for some $t_1 \in \mathbb{C}$. Naturally φ_1 is defined on the disc $B_\epsilon(t_1)$ of radius ϵ about $t_1 \in \mathbb{C}$. Next consider a point t_2 in the boundary of $B_\epsilon(t_1)$. Modulo reducing ϵ from the beginning, we can without loss of generality suppose that $\varphi_1(t_2) = p_2$. However p_2 must belong to K so that the solution φ_2 of X satisfying $\varphi_2(t_2) = p_2$ is defined on the disc $B_\epsilon(t_2)$ of radius ϵ about t_2. Since the union $B_\epsilon(t_1) \cup B_\epsilon(t_2)$ is simply connected, it follows from the monodromy theorem that the solutions φ_1, φ_2 can be patched together into a solution of X defined on $B_\epsilon(t_1) \cup B_\epsilon(t_2)$. Actually, since t_2 is arbitrary in the boundary of $B_\epsilon(t_1)$, it follows that this boundary can be covered by finitely many discs as indicated above. By repeatedly using the monodromy theorem, we conclude that the initial solution φ_1, $\varphi_1(t_1) = p_1$, can be extended to a disc of radius $\epsilon + \delta$ about t_1 with $\delta > 0$ uniformly chosen. An obvious induction shows that this solution is defined on the disc of radius $\epsilon + k\delta$ about t_1 for every $k \in \mathbb{N}$. In other words, φ_1 is defined on all of $\mathbb{C}$.

The desired contradiction follows now from Liouville Theorem: since φ_1 is a bounded holomorphic map from $\mathbb{C}$ to $\mathbb{C}^n$ it must be constant. The proposition is proved. ■

1.4 Basic tools

1.4.1 The Blow-up of manifolds and of foliations

At first sight, the "blow-up" may be thought as a device that simply creates new manifolds out of previous ones. However, it will be seen that it is particularly useful to understand the behavior of foliations or vector fields at singular points.

Definition 1.6 *The blow-up of $\mathbb{C}^2$ at $(0,0)$ is a complex manifold $\widetilde{\mathbb{C}}^2$ obtained by identifying two copies of $\mathbb{C}^2$ in the following way:*

$$(x,t) \simeq (s,y) \Leftrightarrow \; s = \frac{1}{t}; \;\; y = tx \;\; (t \neq 0 \, , s \neq 0) \, ,$$

where (x,t) and (s,y) are the coordinates of the two above mentioned copies.

By definition, the exceptional divisor of $\widetilde{\mathbb{C}}^2$ is $E \subset \widetilde{\mathbb{C}}^2$ given by $\{x=0\}$ (resp. $\{y=0\}$) in the coordinates (x,t) (resp. (s,y)). Therefore E is well-defined and it is isomorphic to $\mathbb{CP}(1)$.

The *blow-up mapping* $\pi : \widetilde{\mathbb{C}}^2 \to \mathbb{C}^2$ is given by $\pi(x,t) = (x,tx)$ and $\pi(s,y) = (sy,y)$. Moreover, it verifies the following:

- $\pi^{-1}(0,0) = E$;
- $\pi : \widetilde{\mathbb{C}}^2 \setminus E \to \mathbb{C}^2 \setminus \{(0,0)\}$ is a holomorphic diffeomorphism;
- π is proper (i.e. the preimage of a compact set is also compact).

Now we shall define the blow-up of a complex 2-dimensional manifold M at a point $p \in M$. Consider a local coordinate chart $\psi : U \to W \subset \mathbb{C}^2$ defined on a neighborhood U of p and such that $\psi(p) = (0,0)$.

Let $\widetilde{W} = \pi^{-1}(W)$, where π is the blow-up mapping. Let M' be the disjoint union of $M \setminus \{p\}$ with $\widetilde{W}$, and consider the following equivalence relation:

$$q_0 \simeq q_1 \Longleftrightarrow q_0 \in U \setminus \{p\}\ , \quad q_1 \in \widetilde{W} \setminus E \ \text{ and } \ \mathrm{q}_1 = \pi^{-1}(\psi(\mathrm{q}_0))\ .$$

The blow-up $\widetilde{M}$ of M at p is the quotient $M'/\simeq$. Notice that $\widetilde{M}$ is indeed a smooth complex manifold for $\widetilde{W}$ is a manifold and $\pi^{-1} \circ \psi : U \setminus \{p\} \to \widetilde{W} \setminus E$ is a holomorphic diffeomorphism.

Similarly, there is a blow-up mapping from $\widetilde{M}$ to M (which will also be denoted by π) that is proper and takes E to p, i.e. $\pi(E) = p$. Moreover, $\pi : \widetilde{M} \setminus E \to M \setminus \{p\}$ is a holomorphic diffeomorphism.

The blow-up of a foliation or vector field can also be defined in a natural way. Let $X = F\frac{\partial}{\partial x} + G\frac{\partial}{\partial y}$ be a vector field on a neighborhood U of the origin in $\mathbb{C}^2$, where F and G are holomorphic functions. Suppose that $(0,0)$ is a singularity of X of order k (k is the minimum between the orders of F and G at $(0,0)$).

Let $F = \sum_{n=k}^{\infty} F_n$, and $G = \sum_{n=k}^{\infty} G_n$, where F_n and G_n are the homogeneous components of degree n of the Taylor series of F and G, respectively.

By using the blow-up mapping $\pi : \widetilde{\mathbb{C}}^2 \setminus E \to \mathbb{C}^2 \setminus \{(0,0)\}$ in the (x,t) coordinates, namely $\pi(x,t) = (x,tx)$ $(x \neq 0)$, π^*X has a natural meaning. In fact, one has

$$\pi^*X = \begin{pmatrix} 1 & 0 \\ -t/x & 1/x \end{pmatrix} \begin{pmatrix} x^k F_k(1,t) + x^{k+1}(F_{k+1}(1,t) + xF_{k+2}(1,t) + \ldots) \\ x^k G_k(1,t) + x^{k+1}(G_{k+1}(1,t) + xG_{k+2}(1,t) + \ldots) \end{pmatrix} .$$

Setting $f(x,t) = (F_{k+1}(1,t) + xF_{k+2}(1,t) + \ldots)$ and $g(x,t) = (G_{k+1}(1,t) + xG_{k+2}(1,t) + \ldots)$, the above equation becomes

$$\pi^*X = x^k[F_k(1,t) + xf(x,t)]\,\frac{\partial}{\partial x} + x^{k-1}[-tF_k(1,t) - xtf(x,t) + G_k(1,t) + xg(x,t)]\,\frac{\partial}{\partial t}\ . \tag{1.4}$$

It is to be noted that this vector field admits a holomorphic extension $\widetilde{X}$ to E ($\{x=0\}$). Thus, $\widetilde{X}$ is defined to be the *blow-up* of X at the singular point $(0,0)$. Moreover, if $k \geq 2$ then $\widetilde{X}$ is singular at every point of E. Similarly, the *blow-up* of the foliation $\mathcal{F}$ associated to X is the foliation $\widetilde{\mathcal{F}}$ associated to $\widetilde{X}$.

The behavior of $\widetilde{\mathcal{F}}$ (and of $\widetilde{X}$) on a neighborhood of E is significantly different, depending on whether or not $G_k(1,t) - tF_k(1,t)$ is identically zero. Let us analyze each case separately.

- If $G_k(1,t) - tF_k(1,t)$ is *not* identically *zero.*

  Dividing Equation (1.4) by x^{k-1}, the foliation remains unchanged and $\widetilde{\mathcal{F}}|_{\widetilde{\mathbb{C}}^2 \setminus E}$ is given by

  $$x[F_k(1,t) + xf(x,t)] \frac{\partial}{\partial x} + [-tF_k(1,t) - xtf(x,t) + G_k(1,t) + xg(x,t)] \frac{\partial}{\partial t} .$$

  Thus, the singularities of $\widetilde{\mathcal{F}}$ on E are determined by the equation $G_k(1,t) - tF_k(1,t) = 0$. Furthermore, we see that E ($\{x=0\}$) is invariant under the foliation in question.

- If $G_k(1,t) - tF_k(1,t) \equiv 0$ (equivalently, if $F_k(x,y)\frac{\partial}{\partial x} + G_k(x,y)\frac{\partial}{\partial y}$ is a multiple of the radial vector field $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$)

  In this case, we may divide Equation (1.4) by x^k so as to obtain:

  $$[F_k(1,t) + xf(x,t)] \frac{\partial}{\partial x} + [-tf(x,t) + g(x,t)] \frac{\partial}{\partial t} .$$

  Hence

  $$\widetilde{\mathcal{F}}|_E = F_k(1,t) \frac{\partial}{\partial x} + [-tF_{k+1}(1,t) + G_{k+1}(1,t)] \frac{\partial}{\partial t} .$$

  Notice that $F_k(1,t)$ is *not* identically *zero*, for if it were, then $G_k(1,t) \equiv 0$ and $(0,0)$ would not be a singularity of order k as assumed from the beginning. Hence, the exceptional divisor E is not preserved by $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}$ may have no singularity contained in E.

The leaves of $\widetilde{\mathcal{F}}$ are transverse to E and are projected by π onto curves passing through $(0,0)$ which are obviously invariant by $\mathcal{F}$. Due to the fact that π is proper, the projection is an analytic set (the common zeros of a finite number of holomorphic functions). A local analytic curve invariant by a foliation and containing the singularity of $\mathcal{F}$ is called a *separatrix* of $\mathcal{F}$. A singularity possessing infinitely many separatrizes is called *dicritical.* These singularities will be characterized in Chapter 2 in connection with Seidenberg's theorem.

1.4.2 Intersection numbers:

Let M be a real 4-dimensional compact oriented manifold. Consider two submanifolds S_1, S_2 of real dimension 2 with the induced orientation.

Recall that the (set-theoretic) intersection of S_1, S_2 consists of a finite number of points provided that S_1 is transverse to S_2 (we also say that S_1, S_2 are in general position). If $p \in M$ is a point of $S_1 \cap S_2$, which is necessarily isolated, then the corresponding tangent space of M splits into a direct sum

$$T_pM = T_pS_1 \oplus T_pS_2\,,$$

where T_pS_1 (resp. T_pS_2) stands for the tangent space to S_1 (resp. S_2) at p. In particular the orientations of T_pS_1 and T_pS_2 taken together yield an orientation for T_pM which may or may not coincide with the original orientation of T_pM. We let $\nu(p) = 1$ if these two orientations coincide and we let $\nu(p) = -1$ otherwise. With these notations we define the *intersection number* $\sharp(S_1 \cap S_2)$ as

$$\sharp(S_1 \cap S_2) = \sum_{p_i \in S_1 \cap S_2} \nu(p_i)$$

provided that S_1, S_2 are in general position. In the general case, we perturb S_1 into S_1' so that S_1', S_2 are in general position and then set $\sharp(S_1 \cap S_2) = \sharp(S_1' \cap S_2)$. The existence of the perturbation S_1' and the fact that $\sharp(S_1' \cap S_2)$ does not depend on the perturbation chosen are standard basic facts of Differential Topology (see for instance [Ko]). When $S_1 = S_2 = S$, the number $\sharp(S_1 \cap S_2)$ is called the *self-intersection* of S.

Remark 1.3 Assume for a moment that M, S_1, S_2 are all complex manifolds (and that S_1, S_2 are submanifolds). If S_1, S_2 are transverse and $p \in S_1 \cap S_2$, then the fact that complex manifolds have a preferred orientation implies that the orientations of M, S_1, S_2 at p are all compatible in the sense that $\nu(p) = 1$. Thus, if S_1, S_2 are complex submanifolds in general position, one automatically has $\sharp(S_1 \cap S_2) \geq 0$.

Nonetheless if S_1, S_2 are not in general position, then it is possible to have $\sharp(S_1 \cap S_2) < 0$. In fact, to obtain a perturbation S_1' of S_1 in general position with S_2, it may be necessary to work in the differential category, i.e. the perturbed submanifold S_1' need no longer be a complex submanifold.

A standard topological argument ensures that $\sharp(S_1 \cap S_2)$ depends only on the *homology class* of S_1, S_2. In the present case, this gives rise to a symmetric *pairing*

$$\langle\ ,\ \rangle\,:\, H_2(M, \mathbb{Z}) \times H_2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

which can be extended by *linearity* to an bilinear form on $H_2(M, \mathbb{R})$ called the *intersection form* of M. In particular the *signature* of the intersection form is an important invariant of a differentiable 4-manifold.

Let us close this short discussion with two simple facts whose verification is left to the reader. They are going to be freely used in the course of this text.

FACT: The self-intersection of the exceptional divisor $\pi^{-1}(0)$ of $\widetilde{\mathbb{C}}^2$ is equal to -1.

FACT: Suppose that S is a compact Riemann surface $S \subset M$ realized as a submanifold on a complex surface M. Suppose that $\widetilde{M}$ is the blow-up of M at a point $p \in S \subset M$. Then, if $\widetilde{S}$ denotes the proper transform of S, one has

$$\sharp(\widetilde{S} \cap \widetilde{S}) = \sharp(S \cap S) - 1 \, .$$

1.4.3 Complex and holomorphic vector bundles:

Assume that M is a compact differentiable manifold. A C^∞*-complex vector bundle on* M consists of a family of *complex vector spaces* $\{E_x\}_{x \in M}$ parametrized by M along with a C^∞-manifold structure on $E = \bigcup_{x \in M} E_x$ such that:

1. The projection map $\pi : E \longrightarrow M$ taking E_x to $x \in M$ is a C^∞-map.
2. For every $x_0 \in M$, there exists an open neighborhood $U \subset M$ of x_0 and a diffeomorphism
$$\varphi_U : \pi^{-1}(U) \longrightarrow U \times C^k$$
commuting with π_1, namely $\pi_{|E_x} = \pi_1 \circ \varphi_{U|E_x}$ or $\pi_1 \circ \varphi_U(E_x) = \text{const}$.
3. The diffeomorphism φ_U considered above takes E_x isomorphically onto $\{x\} \times \mathbb{C}^k$.

The integer k is said to be the *rank* of the vector bundle and the diffeomorphisms φ_U considered above are called its *local trivializations*. Given two trivializations φ_U, φ_V of the same vector bundle, the map

$$g_{UV} \; : \; U \cap V \longrightarrow \mathrm{GL}\,(k, \mathbb{C})$$

given by $g_{UV}(x) = (\varphi_U \circ \varphi_V^{-1})_{|\{x\} \times \mathbb{C}^k}$ is differentiable. In addition these maps satisfy the identities

$$g_{UV}(x) \, . \, g_{VU}(x) = \mathrm{Id} \tag{1.5}$$

$$g_{UV}(x) \, . \, g_{VW}(x) \, . \, g_{WU}(x) = \mathrm{Id} \, , \tag{1.6}$$

where Id stands for the Identity map and the dot "." for the multiplication of matrices in $\mathrm{GL}\,(k, \mathbb{C})$. The functions g_{UV} are called the *transition functions* of the complex vector bundle.

Conversely, to *specify* a structure of complex vector bundle on M, all we need is an open covering $\mathcal{U} = \{U_\alpha\}$ of M together with C^∞-maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}\,(k, \mathbb{C})$ satisfying the identities (1.5) and (1.6). The description of complex vector bundles by means of transition functions is well-suited to define operations with them. Precisely let $E \to M$ and $F \to M$ be complex vector bundles having transition functions $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$ with respect to the same covering $\mathcal{U} = \{U_\alpha\}$ of M. Denote by k (resp. l) the rank of E (resp. F). We give below some standard examples of new vector bundles constructed from the previous ones.

1. $E \oplus F$ (the direct sum). This vector bundle is determined by the transition functions $j_{\alpha\beta}$ given by

$$j_{\alpha\beta}(x) = \begin{pmatrix} g_{\alpha\beta}(x) & 0 \\ 0 & h_{\alpha\beta}(x) \end{pmatrix} \in \mathrm{GL}\,(\mathbb{C}^k \oplus \mathbb{C}^l) \subset \mathrm{GL}\,(\mathbb{C}^{k+l})\,.$$

2. $E \otimes F$ (the tensor product). The transition functions are $j_{\alpha\beta}(x) = g_{\alpha\beta}(x) \otimes h_{\alpha\beta}(x) \in \mathrm{GL}\,(\mathbb{C}^k \otimes \mathbb{C}^l)$.
3. $\Lambda^r E$ (the exterior power). One sets $j_{\alpha\beta}(x) = \Lambda^r g_{\alpha\beta}(x) \in \mathrm{GL}\,(\Lambda^r \mathbb{C}^k)$.
In particular $\Lambda^k E$ is a *line bundle* given by $j_{\alpha\beta}(x) = \mathrm{Det}\, g_{\alpha\beta}(x) \in \mathrm{GL}\,(\mathbb{C}) \simeq \mathbb{C}^*$. This line bundle is referred to as the *determinant bundle of* E.

Definition 1.7 *A subbundle $F \subset E$ of a bundle E is a collection $\{F_x \subset E_x\}_{x\in M}$ of subspaces of the fibers E_x such that $F = \bigcup_{x\in M} F_x$ is a submanifold of E.*

The condition that $F \subset E$ is a submanifold is equivalent to saying that, for every $x \in M$, there exists a neighborhood $U \subset M$ of x and a trivialization $\varphi_U : E_U \to U \times \mathbb{C}^k$ such that

$$\varphi_{U|F_U} : F_u \to U \times \mathbb{C}^l \subset U \times \mathbb{C}^k\,.$$

Relative to these trivializations, the transition functions g_{UV} of E have the form

$$g_{UV}(x) = \begin{pmatrix} h_{UV}(x) & i_{UV}(x) \\ 0 & j_{UV}(x) \end{pmatrix}.$$

The bundle F will have transition functions h_{UV}. Notice that the maps j_{UV} verify the identities (1.5), (1.6) so that they define themselves a vector bundle on M. This vector bundle, denoted by E/F, is called the quotient bundle of E, F. As vector spaces we have $(E/F)_x = E_x/F_x$, nonetheless there is no natural notion of orthogonality.

It is natural to work out the condition on two sets of transition functions defined w.r.t. the same covering of M so that they define the same vector bundle.

Definition 1.8 *1) A map between vector bundles E, F is a C^∞ map $f : E \to F$ such that $f(E_x) \subseteq E_y$ and $f_x : E_x \to F_x$ is linear. The vector bundle E is said isomorphic to F if there if f such that f_x is an isomorphism for all $x \in M$. A vector bundle isomorphic to the product $M \times \mathbb{C}^k$ is called trivial.*

2) A section δ of the vector bundle $E \to^\pi M$ over $U \subseteq M$ is a C^∞-map $\sigma : U \to E$ such that $\sigma(x) \in E_x$ for all $x \in U$. In other words, is a map $\sigma : U \to E$ such that $\pi \circ \sigma = Id$. The set of sections (resp. local sections) is denoted by $\Gamma(E)$ (resp. $\Gamma_U(E)$).

3) A frame for E over $U \subseteq M$ is a collection of $\sigma_1, \ldots, \sigma_k$ sections of M over U such that $\{\sigma_1(x), \ldots, \sigma_k(x)\}$ is a basis of E_x for all $x \in U$. A frame is essentially the same thing as a trivialization over U.

Now let us assume that M is a *complex manifold.* A *holomorphic vector bundle* $E \to M$ is a complex vector bundle together with a structure of complex manifold on E such that the trivializations

$$\varphi_U \;:\; E_U \longrightarrow U \times \mathbb{C}^k$$

are holomorphic diffeomorphisms (we say that they are holomorphic trivializations). Transition maps $g_{UV} : U \cap V \to \mathrm{GL}\,(k, \mathbb{C})$ are now *holomorphic.* Conversely given holomorphic maps $g_{UV} : U \cap V \to \mathrm{GL}\,(k, \mathbb{C})$ satisfying identities (1.5), (1.6), we can construct a holomorphic vector bundle realizing these maps as its transition functions.

Example 1.2.D: The complex tangent bundle.

Let M be a complex manifold and T_xM the complex tangent space of M at $x \in M$ whose dimension over $\mathbb{C}$ is $2N$. Consider a neighborhood U of x with a coordinate chart $\psi_U : U \to \mathbb{C}^n$. This coordinate induces a map

$$\psi_U^* : T_xM \to T_{\psi_U(x)}\mathbb{C}^n \simeq \mathbb{C}\{\partial/\partial x_j \;,\; \partial/\partial y_j\} \simeq \mathbb{C}^{2n}$$

for each x in U. Hence we have a map

$$\psi_U^* : \bigcup_{x \in U} T_xM \longrightarrow U \times \mathbb{C}^{2n}$$

which gives $TM = \bigcup_{x \in U} T_xM$ the structure of a complex vector bundle called the *complex tangent bundle* of M. Transition functions for TM are

$$j_{UV} = \mathrm{Jac}_{\mathbb{R}}(\psi_U \circ \psi_V^{-1})\,.$$

Now T_xM splits into $T_xM = T_xM' \oplus T_xM''$ where $T_xM' = \mathbb{C}\{\partial/\partial z_j\}$ with $\partial/\partial z_j = \partial/\partial x_j - \imath\partial/\partial y_j$. The collection of T_xM' forms a subbundle of T_xM which is called the *holomorphic tangent bundle* of M. The last bundle has transition functions given by

$$j_{UV} = \mathrm{Jac}_{\mathbb{C}}(\psi_U \circ \psi_V^{-1})\,.$$

Example 1.2.E: $\widetilde{\mathbb{C}}^2 \to \pi^{-1}(0)$.

Recall that the blow-up $\widetilde{\mathbb{C}}^2$ of $\mathbb{C}^2$ can be viewed as a holomorphic line bundle (i.e. a holomorphic vector of rank 1) over the exceptional divisor $\pi^{-1}(0) \simeq \mathbb{CP}\,(1)$. Consider the coordinates (x, t), (s, y) for $\widetilde{\mathbb{C}}^2$ introduced in paragraph 1.2 and set

$$U = \{(x, t) \in \mathbb{C}^2 \;;\; t \neq 0\} \quad \text{and} \quad V = \{(s, y) \in \mathbb{C}^2 \;;\; s \neq 0\}\,.$$

We also have the projection π given in coordinates by $\psi_U(x, t) = (x, tx)$ and $\psi_V(s, y) = (sy, y)$. The transition function for the holomorphic tangent space of $\widetilde{\mathbb{C}}^2$ is the Jacobian matrix of $\psi_V \circ \psi_U^{-1}$ for $ts \neq 0$. Thus we obtain

$$\mathrm{Jac}\,(\psi_V \circ \psi_U^{-1}) = \begin{pmatrix} -1/t^2 & 0 \\ x & t \end{pmatrix}\,.$$

Letting $x = 0$ we obtain in particular the transition function for the tangent and normal bundles of $\pi^{-1}(0)$ which are respectively $-1/t^2$ and t.

Among vector bundles, line bundles (i.e. complex bundles of rank 1) play a prominent role in the study of Complex ODEs as well as in Algebraic Geometry as a whole. For this reason, we are going to say a few words about their classification, further information can be obtained in Chapter 4. To begin with, we note that the set of holomorphic line bundles over a compact complex manifold M form a group under the tensor product. The group structure can be made explicit by considering two line bunles L, L' and a common open covering $\{U_i\}$ for M such that both L, L' are trivial over the U_i's. In particular, for $i \neq j$ and $U_i \cap U_j \neq \emptyset$, we have transition functions g_{ij}, g'_{ij} respectively for L, L'. By definition, the functions g_{ij}, g'_{ij} take values on $\mathbb{C}^*$. Setting $h_{ij} = g_{ij}.g'_{ij}$, it is immediate to check that the functions h_{ij} defined on $U_i \cap U_j$ verify the natural cocycle relations so that they define a new line bundle over M. The line bundle associated to these transition functions h_{ij} is $L \otimes L'$. The resulting group is called the *Picard group* of M and it is denoted by $\mathrm{Pic}\,(M)$.

In general the structure of $\mathrm{Pic}\,(M)$ is rather subtle and varies with the holomorphic structure on M (for a fixed underlying differentiable structure). In particular, if M is in addition *algebraic* then $\mathrm{Pic}\,(M)$ is isomorphic to the group of divisors of M (cf. Chapter 4). On the other hand, the classification of complex line bundles in class C^∞, i.e. in differentiable category, depends only on certain characteristic classes associated to M. These classes are called *Chern classes*. The first Chern class, $c_1(L)$, admits a particularly simple geometric interpretation. We consider a C^∞ section σ of L such that M and $\sigma(M)$ are in general position. The intersection $M \cap \sigma(M)$ is then a codimension 2 compact submanifold of M. Thus it defines an element of the homology of M in codimension 2 with integral coefficients. Finally Poincaré duality yields an element in $H^2(M, \mathbb{Z})$ which is exactly $c_1(L)$.

When M has dimension 1 i.e. it is a Riemann surface, then we have some precise statements. Already, it follows from the preceding that the C^∞ classification of complex line bundles is given by the self-intersection of the null section. In particular they are topologically trivial if and only if the self-intersection equals *zero*. As to the holomorphic classification, we have (cf. [Be])

Lemma 1.3 *Two holomorphic line bundles over* $\mathbb{CP}\,(1)$ *are isomorphic (holomorphically diffeomorphic) if and only if their null-section has the same self-intersection.*

By virtue of Lemma (1.3) above, it is easy to construct models for all line bundles over $\mathbb{CP}\,(1)$. Indeed to construct a line bundle whose null-section has self-intersection $n \in \mathbb{Z}$ we take two copies (x, y), (u, v) of $\mathbb{C}^2$ and identify the points satisfying $u = 1/x$ and $v = x^{-n}y$.

Consider now holomorphic line bundles over an elliptic curve S. More precisely let us restrict our attention to the case of topologically trivial line bundles. Unlike rational curves, these holomorphic line bundles have "moduli". The following is also very well known (cf. [A]).

Lemma 1.4 *Topologically trivial holomorphic line bundles over an elliptic curve S are in one-to-one correspondence with points in $\mathbb{C}^*$.*

1.5 Tubular Neighborhoods

When a (complex) manifold M contains a (complex) submanifold S, it is interesting to analyse the structure of a neighborhood of S in M. This is of much interest for Complex EDOs since they occasionally possess a *compact leaf*. It is then natural to investigate the structure of our equation on a neighborhood of this leaf since it makes a lot of information on the global behavior of the equation accessible to us. In particular, it is often necessary to know the structure of the neighborhood of this leaf as a submanifold. In the real setting, this structure is determined by the normal bundle of S in M as follows from the classical tubular neighborhood theorem whose statement we are going to recall.

Definition 1.9 *Let S be a (complex) submanifold of a (complex) manifold M. A tubular neighborhood of S in M is an open set $V \subset M$ together with a (holomorphic) submersion $\Pi : V \to S$ such that*

- *$V \xrightarrow{\Pi} S$ is a (holomorphic) vector bundle.*
- *$S \subset V$ is naturally identified with the zero section of this vector bundle.*

According to the well-known "tubular neighborhood" theorem, a tubular neighborhood always exist in the differentiable category. Indeed a tubular neighborhood V is isomorphic (as real vector space) to the normal bundle $T_N S$ where the fiber N_x over $x \in S$ is isomorphic to $T_x M / T_x S$. Actually the tubular neighborhood should be thought of as a "realization" of the normal bundle as an open neighborhood of S in M.

In the holomorphic setting a holomorphic tubular neighborhood need not exist in general. This means that there may exist *no holomorphic* diffeomorphism between an neighborhood V of $S \subset M$ and a neighborhood of the zero-section in the normal bundle of S.

Remark 1.4 In principle this may cause some confusion since the expression "tubular neighborhood" is often used to refer to a neighborhood of a submanifold without taking in consideration the existence of any diffeomorphism between the neighborhood in question and a neighborhood of the null section of the corresponding normal bundle. Maybe we should use another word, for example collar neighborhood, to tell apart one situation from the other. Unfortunately, it seems that this "abuse of language" is already well established. Hopefully it will lead to no misunderstanding.

To provide an example where a holomorphic diffeomorphism as above does not exist, it suffices to consider a trivial fibration over the unit disc $D \subset \mathbb{C}$ whose fibers are elliptic curves with different complex structures. Clearly the normal bundle of

the fiber L_0 over $0 \in D$ is trivial and, if a holomorphic tubular neighborhood existed, then the restriction of Π to a fiber L_ε would be a holomorphic diffeomorphism between $L\varepsilon$ and L_0. This is impossible provided that we vary the complex structures of the fibers. Here is an explicit construction.

We consider an elliptic curve Γ of periods 2π and τ. We also consider $\mathbb{C} \times D$ equipped with coordinates (x, y) and impose the identifications

$$(x, y) \simeq (x + 2\pi, y) \simeq (x + \tau, y) .$$

Next let us consider the identifications on $\mathbb{C} \times D$ given by $(x, y) \simeq (x + 2\pi, y) \simeq (x + \tau, \lambda y)$ where $\lambda \in \mathbb{C}^*$. With these identifications $\mathbb{C} \times D$ becomes a surface Σ and $\{y = 0\}$ can be thought of as an embedding of an elliptic curve Γ in Σ. Furthermore a neighborhood of the null section in the normal bundle of Γ in Σ is equivalent to a neighborhood of Γ in the original $\mathbb{C} \times D$. However neighborhoods of Γ in $\mathbb{C} \times D$ and in Σ are not equivalent. Actually even more is true: Σ is a topologically trivial line bundle over Γ. However, if $\lambda \neq e^{2\pi i k}$, Fourier series shows that Γ is a *rigid curve* in Σ (i.e. Γ does not admit a holomorphic deformation) which contrasts with the topological triviality of Σ as bundle over Γ.

It is an equally remarkable fact that, as far as fibrations are concerned, the preceding example is universal in a precise sense. To explain it, let us consider a C^∞-trivial fibration whose fibers are complex manifolds (as well as the basis and the projection). More generally the reader may consider a smooth family of compact complex manifolds, i.e. a triplet consisting of a pair of complex spaces $\mathcal{X}$ and $\mathcal{S}$ along with a proper holomorphic map Π form $\mathcal{X}$ onto $\mathcal{S}$ with connected fibers and *flat*. The condition of *flatness* of Π can be expressed by saying the ring of germs of holomorphic functions at every point $x \in \mathcal{X}$ is a module over the ring of germs of holomorphic functions on $\mathcal{X}$ at $\Pi(x)$. If the spaces $\mathcal{X}$, $\mathcal{S}$ are smooth, this condition simply means that Π is a submersion. The "universal" character of our examples results from the following theorem.

Theorem 1.2 (**Fisher & Grauert**) *A fibration as above is locally holomorphically trivial if and only if all fibers are isomorphic as complex manifolds.* □

As a matter of fact, it appears that the existence of tubular neighborhoods for complex submanifolds was not very intensively studied. For the case of Riemann surfaces, a remarkable theorem due to Grauert gives a sufficient condition for the existence of a holomorphic tubular neighborhood.

Theorem 1.3 (**Grauert**) *Assume that S is a Riemann surface embedded as complex submanifold of a complex surface M. If the self-intersection of S is strictly negative, then there exists a holomorphic tubular neighborhood for S in M.*

Before discussing further Theorem (1.3), let us state a useful corollary which was already known to the Italian geometers. If S is a Riemann surface in a complex surface M, we say that S is *contractible* if there exists another complex surface N and a proper holomorphic map $\pi : M \to N$ which collapses S into a point $p \in M$ and induces a holomorphic diffeomorphism from $M \setminus S$ to $N \setminus \{p\}$.

Corollary 1.2 *Let S be a rational curve in M and assume that S has self-intersection -1. Then S is contractible.*

Proof. Because of Grauert's theorem, a neighborhood of S in M is equivalent to a neighborhood of $\pi^{-1}(0)$ in $\widetilde{\mathbb{C}}^2$. It is then obvious that S can be collapsed. ■

Let us now make some comments concerning Grauert's theorem or, more generally, the existence of tubular neighborhoods for compact Riemann surfaces S embedded in complex surface M.

To begin with it is necessary to point out the natural role played by the assumption on the negativity of the self-intersection of S. This lies in the fact that a Riemann surface S with negative self-intersection in M is *isolated* in the sense that there is a neighborhood U of $S \subset M$ containing no embedded compact Riemann surface other than S itself. Otherwise, we may consider a Riemann surface S' contained in a sufficiently small neighborhood U as a *deformation of* S, i.e. they belong to the same class in the second homology group of M. Hence we would have $S.S = S.S'$. However $S.S'$ is nonnegative as the intersection number of two complex submanifolds and the resulting contradiction shows that S is isolated in the mentioned sense. Clearly for an isolated Riemann surface S, the mechanism we have used before to exclude the existence of a tubular neighborhood (as in Definition 1.9) for S can no longer apply: we cannot find in U other Riemann surfaces with complex structure different from that of S.

A second comment involving Theorem (1.3) is that it suggests a rather naive way to study the case of nonnegative self-intersection. This consists of blowing up a number of points in S so as to bring the self-intersection of S to, say, -1 (where S is naturally identified with its proper transform). The effect of these blow-ups amounts to adding a finite number of exceptional rational curves having normal crossings with S. In the new manifold, S has negative self-intersection and thus a neighborhood of it is isomorphic to a neighborhood of the null section in the normal bundle. To establish the existence of a tubular neighborhood for the "initial" S is then equivalent to show that the isomorphism in question can be extended to a neighborhood of the exceptional curves in a natural way. Although naive, this remark is useful in some cases, especially if one wants to prove index theorems.

1.6 Miscellaneous

This section is intended to collect several results, some of them very non-trivial, that are needed for a better understanding of the material discussed in the subsequent chapters. The discussion below is by no means self-contained. In fact, at some points it is necessary to use notions that are introduced only in Chapter 3, this is of course compounded with the several results presented without proof. To remedy for this situation, we try to keep the following two botton lines:

1. Precise references for the theorems stated without proofs.

2. Whenever we state a theorem using notions and/or terminology that will be introduced only later, we also state the most used especial versions of it in a simpler language that hopefully can immediately be "decoded".

The section is organized in three main families of results: we begin with results belonging to Complex Analysis more directly. The second type of results discussed revolves around Serre's GAGA theorems. Finally we shall discuss the nature of automorphisms of compact complex manifolds.

1.6.1 Local results in Complex Analysis

We are going to begin with the well-known Weierstrass preparation and division theorems for germs of holomorphic functions. The corresponding proofs can be found in any standard book about several complex variables.

Theorem 1.4 (**Weierstrass Preparation Theorem**) *Let f be a holomorphic function defined on a neighborhood U of the origin of $\mathbb{C}^n$ having the form $U = U' \times \{|z_n| < r\}$ for some $r > 0$ and where U' is a neighborhood of the origin in $\mathbb{C}^{n-1}$. Suppose also that $f(0, z_n)$ is not identically zero on the disc $|z_n| < r$. Suppose also that $f(0, z_n)$ does not have zeros on the circle of radius r and let k denote the number of its zeros, counted with multiplicities, in the disc of radius r. Then, in a suitable neighborhood $V = V' \times \{|z_n| < r\} \subset U$, we have*

$$f(z', z_n) = u(z', z_n).(z_n^k + c_1(z')z_n^{k-1} + \cdots + c_k(z'))$$

where $z' = (z_1, \ldots, z_{n-1})$, $u(0, \ldots, 0) \neq 0$ and c_i is holomorphic for $i = 1, \ldots, k$.

It is convenient to say that a *Weierstrass polynomial* of degree k in z_n is a monic polynomial of degree k in z_n having coefficients that are holomorphic functions of $z' = (z_1, \ldots, z_{n-1})$. Here these coefficients, other than the leading one, are supposed to vanish at the origin. Summarizing, a Weierstrass polynomial as above is a function of the form

$$z_n^k + c_1(z')z_n^{k-1} + \cdots + c_k(z')\,,$$

with c_i holomorphic satisfying $c_i(0) = 0$, $i = 1, \ldots, k$.

The companion result of Theorem (1.4) is the so-called Weierstrass Division Theorem stated below:

Theorem 1.5 (**Weierstrass Division Theorem**) *Let f be as before. Consider a Weierstrass polynomial h of degree k in z_n. Then f can uniquely be written in the form $f = gh + q$ where g is holomorphic on a neighborhood of the origin in $\mathbb{C}^n$ and q is a polynomial in z_n of degree strictly less than k.*

Recall that an *analytic subvariety* V of a complex manifold M is a subset which is locally given as the intersection of the zero-set of finitely many analytic functions. This means that if $p \in V$, there is a neighborhood $U \subset M$ containing p together

with holomorphic functions $g_1, \ldots, g_r$ defined on U such that $V \cap U = \bigcap_{i=1}^{r} g_i^{-1}(0)$. It is clear that if we have an holomorphic map f between complex manifolds M, N, the preimage by f of an analytic subvariety $V \subset N$ is itself an analytic subvariety of M. This is however not true for *direct images* unless the map in question is, in addition, proper. Indeed this is the contents of the *Proper Mapping Theorem*, also called Remmert's mapping theorem, whose statement is as follows.

Theorem 1.6 *Consider complex manifolds M, N along with a proper holomorphic map $f : M \to N$. If $V \subset M$ is a closed analytic subset of M, the its image $f(V) \subset N$ is closed as well.* $\square$

It is also relevant to have some feeling concerning the local nature of an analytic subvariety (also called "an analytic set"). In the sequel we present a rather elementary and yet useful way to "parametrize" one given set. The procedure is very standard and is detailed in several textbooks. Since we are dealing with local parametrizations, it suffices to consider analytic sets defined on a neighborhood of the origin in $\mathbb{C}^n$ which will still be denoted by V.

Let us begin with the case of *curves*, i.e. analytic sets of dimension 1 (the dimension of a singular set is the maximal dimensional of the complement of the singular set). Note also that the singular set must have codimension at least one inside the analytic set in question. In particular for the case of (germs of) analytic curves, we can suppose that the origin is the unique singularity of it. Germs of curves are known to admit a *Puiseaux parametrization* (cf. for example [Fu]). Precisely there exists a holomorphic map $P : \mathbb{D} \to V$, where $\mathbb{D} \subset \mathbb{C}$ stands for the unit disc, satisfying the conditions below.

- $P(0) = (0, \ldots, 0)$ and P is one-to-one from $\mathbb{D}$ to V.
- P is a diffeomorphism from $\mathbb{D}^*$ to $V \setminus \{(0, \ldots, 0)\}$.

In terms of a coordinate $t \in \mathbb{D}$, we can set

$$P(t) = (t^k, P_2(t), \ldots, P_n(t)) .$$

Unfortunately, the existence of the Puiseaux parametrization does not generalize to higher dimensional analytic sets. Parametrizing these sets is therefore a less precise procedure that essentially amounts to noticing that they (locally) are ramified coverings of a plane of suitable dimension. We describe below only the associated geometric picture. Proofs for our statements can easily be produced with the help of Weierstrass Preparation Theorem.

1.6.2 Introduction to GAGA theorems and to Hartogs principle

Let us now move to the second topic to be discussed in the section, namely the GAGA theorems [Se]. In vague but meaningful terms, one states the *GAGA principle* by saying that *any analytic object defined over an algebraic manifold is indeed algebraic.*

To begin the term *algebraic variety* will be used in what follows to refer to an irreducible Zariski-closed subset of some complex projective space $\mathbb{CP}(n)$. Recall that a Zariski-closed subset of $\mathbb{CP}(n)$ is nothing but the common zero-set of a finite collection of homogeneous polynomials in the variables $(x_0, x_1, \ldots, x_n)$. It should be pointed out here that Hilbert Basis Theorem tells us that one does not obtain a "larger class" of sets by allowing infinite collections of these polynomials. Finally the term "irreducible" means that the ideal of regular functions vanishing identically on this set forms a prime ideal of the ring of regular functions on $\mathbb{CP}(n)$. As usual, by an *algebraic manifold* it is meant a *smooth* algebraic variety. The first instance of "GAGA principle" is manifested by the celebrated Chow Lemma.

Theorem 1.7 (**Chow Lemma**) *An analytic subvariety of a projective space is algebraic.*

On algebraic varieties one can consider the ring of regular functions as well as its fraction field called the field of rational functions on M. It is well-known that a rational function on V is given in (say inhomogeneous) coordinates as the quotient of two polynomials. A rational function is said to be regular at $p \in V$, if it can be represented as a quotient of polynomials with denominator different from *zero* at p. A regular function is nothing but a rational function that is regular at every point of V. Finally a rational (resp. regular) map is simply a map whose coordinates are rational (resp. regular) functions. In particular, a rational map between algebraic varieties $V_1 \subset \mathbb{CP}(n)$ and $V_2 \subset \mathbb{CP}(m)$ is given by the restriction of a rational map from $\mathbb{CP}(n)$ to $\mathbb{CP}(m)$. All this is fairly basic material in algebraic geometry that can be found in any textbook, for example [Sh1, Sh2].

The second fundamental instance of GAGA principle can then be stated as

Theorem 1.8 *Every meromorphic function defined on an algebraic variety is rational.*

Nice treatments of the above statements are given in [G-Ha] and in [Mu]. The proof in [Mu] has the advantage of being rather self-contained while the proof in [G-Ha] is shorter, however it relies on the Proper Mapping Theorem.

1.6.3 The automorphism group of complex manifolds

Finally let us briefly review some basic facts about automorphism groups of a compact complex manifold. The first very general and important result is:

Theorem 1.9 *The automorphism group of a compact complex manifold M is a complex Lie group of finite dimension acting holomorphically on M.*

A proof of this result can be found in the nice survey article of B. Deroin [D]. It is based on a couple of very important theorems whose interest goes beyond Theorem 1.9. These are as follows.

Theorem 1.10 (**Cartan & Serre**, [Ca-Se]) *Let E be a (finite dimensional) holomorphic vector bundle over a compact complex manifold M. Then the vector space $\Gamma(M, E)$ consisting of holomorphic sections of E has finite dimension.*

Theorem 1.11 (**Bochner & Montgomery**, [B-Mo]) *Let M be as before. Then every automorphism of M sufficiently C^0-close to the identity is the time-one map induced by a holomorphic vector field on M.*

Suppose we want to determine the connected component containing the identity of the group of automorphism of a complex manifold M. An equivalent formulation for this problem is to describe the Lie algebra of this automorphism group denoted by $\mathrm{Aut}\,(M)$. We can starting thinking of the n-dimensional complex projective space $\mathbb{CP}(n)$. It is well-known that the corresponding automorphism group is $\mathrm{PSL}\,(n+1, \mathbb{C}) = \mathrm{PGL}\,(n+1, \mathbb{C})$ and the associated action is nothing but the projectivization along radial lines of the linear action of the invertible matrices of $\mathrm{GL}\,(n+1, \mathbb{C})$ on $\mathbb{C}^{n+1}$. As a motivation for our discussion, and in particular for the classical Blanchard Lemma, let us sketch a proof of this fact. First of all, it is clear that $\mathrm{PSL}\,(n+1, \mathbb{C})$ is contained in $\mathrm{Aut}\,(\mathbb{CP}(n))$. If g is an algebraic automorphism automorphism of $\mathbb{CP}(n)$ then it is represented in affine coordinates by algebraic maps, ie quotient of polynomials. Imposing that these expressions must glue together by change of coordinates is a very strong condition and allows us to show that g coincides with the action of an element of $\mathrm{PSL}\,(n+1, \mathbb{C})$.

The above discussion did not totally solve the problem since there might exist *nonalgebraic* automorphisms. Naturally the GAGA theorems allows to rule out this possibility and thus to actually establish that $\mathrm{Aut}\,(\mathbb{CP}(n))$ coincides with $\mathrm{PSL}\,(n+1, \mathbb{C})$. However Blanchard Lemma provides with a more elementary alternative that does not rely on GAGA Principle. First we state Blanchard Lemma. Here is convenient to recall that a *fibration* on a compact complex manifold M is nothing but a holomorphic map $\mathcal{P} : M \to N$ onto another compact complex manifold of smaller dimension that is a submersion at generic points. If $S \subset M$ stands for the singular values of $\mathcal{P}$, then the well-known theorem of Ereshemann implies the existence of a *locally trivial differentiable fibration* of $M \setminus \mathcal{P}^{-1}(S)$ over $N \setminus S$. Now we have:

Theorem 1.12 (**Blanchard Lemma**) *Consider a fibration as above in a complex compact manifold M. Then every holomorphic vector field globally defined on M is such that its flow sends fibers of $\mathcal{P}$ to fibers of $\mathcal{P}$.*

Proof. The theorem is of local nature so that we consider a open set U of M fibering over unit polydisc of some $\mathbb{C}^k$. Note that every vector field as in the statement is complete since M is compact. Next let φ^t denote the induced flow for $t \in \mathbb{C}$. Denote by F_0 the fiber sitting over the origin of $\mathbb{C}^k$. If t is sufficiently small in absolute value, it follows that the image $\varphi^t(F_0)$ is still contained in U. It can therefore be composed with the projection from U to the unit polydisc. The resulting map is holomorphic from F_0 to the polydisc. Because F_0 is compact, this map must be constant. In

other words, φ^t is contained in a fiber of $\mathcal{P}$. The statement promptly follows from this observation. ■

Since $\mathrm{Aut}\,(M)$ is a Lie group of finite dimension, its connected component $\mathrm{Aut}_0(M)$ containing the identity is obtained through the corresponding Lie algebra. Precisely, the elements of $\mathrm{Aut}_0(M)$ are induced by the holomorphic vector fields carried by M. In this context, the upshot of Blanchard Lemma is that, by construting fibrations in our manifolds, the problem of identifying holomorphic vector fields on M can be reduced to smaller dimensional cases. In particular, if M is algebraic, then fibrations can always be constructed by considering the "level hypersurfaces" of a nonconstant meromorphic function on M. Strictly speaking these hypersurfaces define a pencil, rather than a fibration, on M. Indeed, a meromorphic function is in general not defined on a subvariety of codimension 2. Desingularization results however allow us to easily turn this pencil in an actual fibration.

In the special case of $\mathbb{CP}(n)$, we can consider the meromorphic function given in affine coordinates by x_2/x_1. Clearly this function is not defined on the subvariety $\{x_2 = x_1 = 0\}$. Nonetheless, we blow this submanifold up to produce a new manifold M where it is defined a fibration whose basis is $\mathbb{CP}(1)$ and whose typical fiber is $\mathbb{CP}(n-1)$. Thanks to Blanchard Lemma, holomorphic vector fields on M can be projected over the basis to define a vector field on $\mathbb{CP}(1)$. When this projection is trivial, the vector field must be tangent to the fibers and, in particular, it has to induce a holomorphic vector field on $\mathbb{CP}(n-1)$.

To start out the recurrence procedure, it is therefore enough to determine the automorphism group of $\mathbb{CP}(1)$. This case can easily be handed: it suffices to consider the stereographic projection of $\mathbb{CP}(1)$ to $\mathbb{C}$ and apply, for example, Picard theorem to conclude that an automorphism of $\mathbb{C}$ must have an "algebraic nature" at infinity. With this information in hand, the recurrence procedure described above becomes effective. In particular, for $\mathbb{CP}(2)$, where the blown-up manifold M coincides with the first Hirzebruch surface F_1, the reader can check [Ak] for explicit calculations.

It remains the problem of passing from $\mathrm{Aut}_0(M)$ to $\mathrm{Aut}\,(M)$. This is more elaborate. In the case of $\mathbb{CP}(n)$ we can exploit the size of $\mathrm{Aut}_0(\mathbb{CP}(n))$ to show that $\mathrm{Aut}\,(\mathbb{CP}(n))$ actually coincides with $\mathrm{Aut}_0(\mathbb{CP}(n))$. Let us briefly sketch the argument. Suppose that f is an element of $\mathrm{Aut}\,(\mathbb{CP}(n)) \setminus \mathrm{Aut}_0(\mathbb{CP}(n))$. Then f naturally acts on the Lie algebra of holomorphic vector fields of $\mathbb{CP}(n)$. We claim the existence of an eigenvector for this action, ie. there is a holomorphic vector field X on $\mathbb{CP}(n)$ such that $f^*X = cX$ where $c \in \mathbb{C}$. The proof amounts to observe that $\mathrm{Aut}_0(M) \simeq \mathrm{PSL}\,(n+1, \mathbb{C})$ acts transitively on $n+2$-tuples of points. Hence we can suppose that f has, say, $n+1$ fixed points. Besides the Lie subalgebra of vector fields having singularities at the $n+1$ fixed points of f is generated by a single vector field X. Thus we must have $f^*X = cX$ as desired. To finish the proof, we observe that X has a first integral, i.e. its orbits can naturally be compactified into rational curves in $\mathbb{CP}(n)$. By construction f must preserve the corresponding pencil of rational curves so that we can now apply a recurrence procedure to obtain a contradiction with the assumption that f belongs to $\mathrm{Aut}\,(\mathbb{CP}(n)) \setminus \mathrm{Aut}_0(\mathbb{CP}(n))$.

Even in dimension 2, an automorphism of an algebraic surface can have a “rich dynamics”, in particular, they need not preserve any fibration on M. Clearly these diffeomorphisms cannot belong to $\mathrm{Aut}_0(M)$. There is however a simple way to measure the chances of an automorphism of an algebraic surface M be strictly more complicated than those belonging to $\mathrm{Aut}_0(M)$. In fact, if the corresponding action on the second homology group $H^2(M)$ of M is trivial, then the automorphism must preserve every fibration defined on M. The proof of this claim is given below; we left to the reader the straightforward generalizations of this lemma to higher dimensions.

Lemma 1.5 *Suppose that M is a compact complex surface supporting a fibration $\mathcal{P} : M \to S$ over a Riemann surface S. Let f be an automorphism of M acting trivially on $H^2(M)$. Then f preserves $\mathcal{P}$.*

Proof. We begin by noticing that all the fibers of $\mathcal{P}$ are homologous one to the others. Thus they are associate to a well defined class $[L]$ in $H^2(M)$. Since f acts trivially in $H^2(M)$, we have $f^*[L] = [L]$. On the other hand, since L is a fiber of a fibration, one has $[L].[L] = 0$. However, if f did not preserve $\mathcal{P}$, then the image $f(L)$ of a generic chosen fiber L would intersect other fibers non trivially another fiber, say L'. Since $f(L)$ and L' are both complex submanifolds of M, it would follows that $\sharp(f(L) \cap L') > 0$ (strictly). However the intersection number depends only on the homology class of the submanifold, so we have

$$0 < [f(L)][L'] = [L][L'] = [L][L] = 0\,.$$

The resulting contradiction establishes the lemma. ■

Chapter 2

Singularities of holomorphic vector fields

2.1 Introduction

The basic topic of the present chapter is the study of singularities of complex vector fields in dimension 2. In the preceding chapter we had already traced a parallel between real and complex ordinary differential equations and point out the main aspects in which they differ. For instance, the analogue to the maximal interval of definition for solutions of *real* ODEs does not exist, in general, in the complex case. On the other hand, the geometric idea of viewing the solutions of real ODEs (away from the singular set) as 2-dimensional leaves of a foliation can easily be transported to the complex scenario. In the case of complex ODEs, we can say that the orbits are Riemann Surfaces and that, away from the singularities, they are leaves of a holomorphic foliation.

In Section 2 we give some basic results related to singularities of holomorphic foliations. We begin with the foliation $\mathcal{F}$ associated to a vector field X, having an isolated simple singularity at $(0,0)$, with non-zero eigenvalues λ_1 and λ_2. In other words,

$$X(x_1, x_2) = [\lambda_1 x_1 + \varphi_1(x_1, x_2)] \frac{\partial}{\partial x_1} + [\lambda_2 x_2 + \varphi_2(x_1, x_2)] \frac{\partial}{\partial x_2} \,. \tag{2.1}$$

We discuss the problem of linearizing such vector fields. More precisely, understand under which conditions there exists a holomorphic change of coordinates that linearizes the system. In fact, a *formal* change of coordinates can easily be found, except for very specific resonant cases. What is more challenging is to prove its convergence. It becomes clear that the existence of a holomorphic change of charts linearizing X depends entirely on the values of the eigenvalues. For instance, if λ_1/λ_2 do not belong to $\mathbb{R}_-$ and if neither λ_1/λ_2, nor λ_2/λ_1 belong to $\mathbb{N}$ then, in appropriate coordinates, it is indeed linear. This is the contents of the well-known Poincaré Linearization Theorem. Now if the singularity belongs to the Siegel domain, i.e., $\lambda_1/\lambda_2 \in \mathbb{R}_-$, we cannot find a linearizing holomorphic change of coordinates. Nev-

ertheless, in local coordinates (y_1, y_2), Equation 2.1 may be expressed as

$$X = \lambda_1 y_1[1 + (h.o.t.)]\partial/\partial y_1 + \lambda_2 y_2[1 + (h.o.t.)]\partial/\partial y_2 \,.$$

Next we shall investigate the case in which one of the eigenvalues, say λ_2 is *zero* and $\lambda_1 \neq 0$. Such singularities are called *saddle-nodes*. We obtain a normalization for this type of singularity, known as Dulac's Normal Form. This result simply states that vector fields containing saddle-node singularities may be given in local coordinates (y_1, y_2) by

$$X(y_1, y_2) = [y_1(1 + \lambda y_2^p) + y_2 R(y_1, y_2)] \frac{\partial}{\partial y_1} \; + \; y_2^{p+1} \frac{\partial}{\partial y_2} \,, \tag{2.2}$$

up to an invertible factor.

Subsection 3.2 is devoted to a brief study of singularities in higher dimensions. In particular we give a generalization of Poncaré Linearization Theorem and some results related to saddle-node singularities in dimension 3.

Section 3 is the core of the present text and undoubtedly contains the most important results of this text. It is strongly inspired in the work of J.-F. Mattei and R. Moussu (cf. [M-M]) and J. Martinet and J.-P. Ramis (cf. [Ma-R]). We begin by revisiting saddle-node singularities, but this time from a more advanced standpoint. We are interested in understanding whether there exists a holomorphic change of coordinates where the term $R(y_1, y_2)$ in Equation 2.2 becomes *identically* zero. Even though it is possible to obtain a formal conjugacy between the two normal forms, in general, this application does not converge on a neighborhood of the singularity. However, in certain sectors of the neighborhood, the formal conjugacy is indeed *summable*. This is basically the contents of the Hukuara-Kimura-Matuda Theorem. So the next step is to study the functions that "glue" together these sectors; i.e., the sector changing diffeomorphisms. In the simplest case, there are two diffeomorphisms that realize the two possible changes of sectors, depending on the connected component of the domain of intersection that is being considered. One of them is a translation and the other one happens to be a diffeomorphism tangent to the identity. The interesting issue here, is that these diffeomorphisms are not unique. Only their conjugacy classes is canonic and can be used to parameterize the moduli space of the saddle-nodes. This leads us to a related topic, namely, the classification of diffeomorphisms of the type $f(z) = z + z^2 + \cdots$, following the work of S. Voronin [Vo]. The procedure to be employed is again based on sectorial normalizations, whereas the normalizing maps will now be constructed by means of the Measurable Riemann Theorem.

Up to this point we have only been dealing with simple singularities. One might ask what is to be done about more degenerate singularities. Indeed, in dimension 2 we do not really have to worry about them, since there is an effective method to reduce any singularity to a "superposition" of simple ones. This is precisely what is done in Subsection 4.2, following [M-M]. Essentially, the idea is that by composing a finite number of blow-up applications we reduce a singularity of higher order to an

arrangement of curves containing only simple singularities. This is the contents of the Seidenberg Theorem. Another reduction may yet be done in the case of simple singularities: either it is reduced to a saddle-node singularity or both λ_1/λ_2 and λ_2/λ_1 do not belong to $\mathbb{N}$.

Finally Subsection 4.3 is devoted to Mattei-Moussu Theorem, regarding the existence of holomorphic first integrals for foliations. In their joint work ([M-M]), J.-F. Mattei and R. Moussu find necessary and sufficient conditions for the existence of non-constant holomorphic functions that are constant along the leaves of a foliation $\mathcal{F}$ with an isolated singularity. Moreover these conditions are of topological nature. Their theorem is as follows.

Theorem 2.1 (Mattei-Moussu [M-M]) *Consider the holomorphic foliation $\mathcal{F}$ defined on $U \subset \mathbb{C}^2$ with an isolated singularity at $(0,0)$. Suppose that it satisfies the following conditions:*

1. *Only a finite number of leaves of $\mathcal{F}$ accumulates on $(0,0)$;*
2. *The leaves of $\mathcal{F}$ are closed on $U \setminus \{(0,0)\}$.*

Then $\mathcal{F}$ has holomorphic non-constant first integral $f : U \to \mathbb{C}$.

We have divided the proof of Mattei-Moussu Theorem into 3 parts so as to make the exposition more transparent. The first step is to show that under these assumptions the singularity can be reduced to a superposition of singularities belonging to the Siegel domain. This is done by studying the local holonomy of a leaf with respect to a loop encircling each type of singularity that might be contained in the Seidenberg tree of $\mathcal{F}$. We reach the conclusion that the only way we do not violate Conditions 1 and 2 is if the singularities are in the Siegel domain.

The next step is to show that the singularities in the Seidenberg tree admit local first integrals. This is done by showing that λ_1/λ_2 belong to $\mathbb{Q}_-$ and using the fact (also due to J.-F. Mattei and R. Moussu) that if the holonomy associated to a leaf of $\mathcal{F}$ is linearizable, then the vector field related to this foliation is linearizable as well.

Finally we extend the local first integrals in order to obtain a global one. We show this in the case where all of the singularities are reduced by a single blow-up and the general case follows easily by induction.

2.2 Normal Forms for Singularities of Holomorphic Foliations

This section is devoted to the study of isolated singularities of holomorphic foliations in dimension 2.

Consider a holomorphic foliation $\mathcal{F}$ with an isolated singularity at $(0,0)$. As previously seen, there is a holomorphic vector field X associated to this foliation, which is uniquely defined up to a multiplicative function f such that $f(0,0) \neq 0$.

The *eigenvalues* for the foliation $\mathcal{F}$ at $(0,0)$ are defined to be the eigenvalues λ_1, λ_2 associated to the vector field X at $(0,0)$, i.e. the eigenvalues of $\mathrm{Jac}X(0,0)$. Notice however that the precise values of λ_1, λ_2 are not well defined (except for the case where both are equal to zero) since the representative of the foliation is defined up to a multiplicative unitary function. What is, in fact, well defined for the foliation is the ratio λ_1/λ_2 (assuming $\lambda_2 \neq 0$). The ration is clearly invariant by the choice of vector field associated to $\mathcal{F}$. We have therefore three possibilities:

(a) $\lambda_1 = \lambda_2 = 0$;

(b) $\lambda_1 = 0$, $\lambda_2 \neq 0$;

(c) $\lambda_1 \neq 0$, $\lambda_2 \neq 0$.

which are well-defined in the sense that the eigenvalues of two representatives of $\mathcal{F}$ belong to the same case (a),(b) or (c). A singularity is said to be *simple* if at least one of its eigenvalues is different from zero. If exactly one of its eigenvalues is equal to zero, then the singularity is called a *saddle-node*.

The *order* of a foliation $\mathcal{F}$ at $(0,0)$ is the degree of the first non-vanishing homogeneous component of the Taylor series of X based at $(0,0)$. Naturally, this order does not depend on the chosen vector field X.

We will begin by studying simple singularities, taking into account the problem of linearization. Afterwards the more general case will be dealt with using the Seidenberg's Theorem.

Let X, Y be two holomorphic vector fields defined in a neighborhood of the origin, singular at this point. We say that X is analytically (resp. formally, C^∞, C^k) conjugate to Y if there exists a holomorphic (resp. formal, C^∞, C^k) diffeomorphism H, such that $H(0) = 0$ and

$$Y = (DH)^{-1}(X \circ H).$$

We say that X and Y are analytically (resp. formally, C^∞, C^k) equivalent if X is analytically (resp. formally, C^∞, C^k) conjugate to fY, for some holomorphic function f verifying $f(0) \neq 0$.

By analytic (resp. formal C^∞, C^k) classification we mean a partition of the set of holomorphic vector fields in classes whose elements are analytically (resp. formally, C^∞, C^k) conjugate to the others in the same class. A vector field X is said analytic (resp. formal, C^∞, C^k) linearizable if it belongs to the analytic (resp. formal, C^∞, C^k) class of the vector field $J_0^1 X$ (i.e. the linear part of X).

2.2.1 Vector Fields with Non-Zero Eigenvalues

Let us consider the ODE generated by a holomorphic vector field X with an isolated singularity at $(0,0)$. Suppose further that its eigenvalues at $(0,0)$ are λ_1 and λ_2,

both different from *zero*. In other words, we are considering the foliation associated to the system of ODEs:

$$\begin{cases} \dot{x_1} = \lambda_1 x_1 + \varphi_1(x_1, x_2) \\ \dot{x_2} = \lambda_2 x_2 + \varphi_2(x_1, x_2) \end{cases} . \tag{2.3}$$

where φ_1, φ_2 have order at least 2. By $\dot{x}$ we mean dx/dT. Note that we are assuming $\mathrm{Jac}X(0,0)$ diagonalizable. We now wish to investigate the existence of a formal change of coordinates that linearizes this system. That is, the existence of a formal map H such that $DH.X = J^1_{(0,0)}X \circ H$.

We shall adopt the the following notations. Let $Q = (q_1, q_2)$, $x^Q = x_1^{q_1} x_2^{q_2}$ and $\|Q\| = q_1 + q_2$. Suppose that, under the notations above, φ_1, φ_2 are written in the form

$$\varphi_1 = \sum_{\|Q\|>1} \varphi_{1,Q} x^Q \ \ ; \ \ \varphi_2 = \sum_{\|Q\|>1} \varphi_{2,Q} x^Q .$$

and consider the formal change of coordinates

$$x_1 = y_1 + \zeta_1(y_1, y_2) \quad \text{where} \quad \zeta_1(y_1, y_2) = \sum_{\|Q\|>1} \zeta_{1,Q} y^Q , \tag{2.4}$$

$$x_2 = y_2 + \zeta_2(y_1, y_2) \quad \text{where} \quad \zeta_2(y_1, y_2) = \sum_{\|Q\|>1} \zeta_{2,Q} y^Q . \tag{2.5}$$

Assume that, in these new coordinates, system (2.3) is given by:

$$\begin{cases} \dot{y_1} = \lambda_1 y_1 + \psi_1(y_1, y_2) \\ \dot{y_2} = \lambda_2 y_2 + \psi_2(y_1, y_2) \end{cases} . \tag{2.6}$$

Note that the right hand side corresponds to the linear part of X if and only if ψ_1, ψ_2 vanishes identically.

Substituting (2.4) and (2.5) in (2.3), we obtain the following relations:

$$\sum_{\|Q\|>1} (\delta_{1,Q}\zeta_{1,Q} + \psi_{1,Q}) y^Q = \varphi_1(y_1 + \zeta_1, y_2 + \zeta_2) - \sum_{k=1,2} \frac{\partial \zeta_1}{\partial y_k} \psi_k , \tag{2.7}$$

$$\sum_{\|Q\|>1} (\delta_{2,Q}\zeta_{2,Q} + \psi_{2,Q}) y^Q = \varphi_2(y_1 + \zeta_1, y_2 + \zeta_2) - \sum_{k=1,2} \frac{\partial \zeta_2}{\partial y_k} \psi_k , \tag{2.8}$$

where $\delta_{1,Q} = \lambda_1 q_1 + \lambda_2 q_2 - \lambda_1$ and $\delta_{2,Q} = \lambda_1 q_1 + \lambda_2 q_2 - \lambda_2$. Notice that if $\delta_{i,Q} \neq 0$ $(i = 1, 2)$ we can always find $\zeta_{i,Q}$ such that $\psi_{i,Q} = 0$. This can be seen by induction on $\|Q\|$ and recalling that φ_i has terms of order at least 2. In fact, this last implies that the coefficients of y^Q on the right hand side of Equations (2.7) and (2.8) depend only on terms $\zeta_{i,P}$ for $\|P\| < \|Q\|$. Therefore $\zeta_{i,Q}$ depend on the coefficients of φ_i, on the terms $\zeta_{i,P}$ with $\|P\| < \|Q\|$ and on $\delta_{i,Q}$ (which are non-zero by assumption). More precisely, $\zeta_{i,Q} = \frac{f}{\delta_{i,Q}}$, where f is a function of certain coefficients of φ_i and $\zeta_{i,P}$

for $\|P\| < \|Q\|$. Thus if $\delta_{i,Q} \neq 0$ for all $Q \in \mathbb{N}^2$ then there always exist a formal change of coordinates linearizing X.

This settles the problem of finding a formal change of coordinates. We have therefore that a vector field is formally conjugate to its linear part if there are no *resonances*.

Definition 2.1 *Suppose that the origin is a singular point of a vector field X. Let $\lambda = (\lambda_1, \lambda_2)$ be the vector constituted by the eigenvalues of* $\mathrm{Jac}X(0,0)$. *We say that the eigenvalues are resonant if, for some i, there exists $I = (i_1, i_2) \in \mathbb{N}_0^2$ with $\sum_{j=1}^{2} i_j \geq 2$ such that*

$$\lambda_i = (I, \lambda) = i_1\lambda_1 + i_2\lambda_2 .$$

In this case the monomials $x^I\partial/\partial x_i$, where $x^I = x_1^{i_1}x_2^{i_2}$, are said to be resonant. If $\dim\{m \in \mathbb{Z}^2 : (m,\lambda) = 0\} = k$ *then X is said k-resonant.*

We have just seen that in the absence of resonances there exists a formal diffeomorphism H taking X into its linear part. However, nothing guarantees that H, or equivalently the series $\zeta_i(y_1, y_2)$ $(i = 1, 2)$, do indeed converge. In fact this may not occur.

Before dealing with the problem of convergence, we shall introduce some useful notations. Given a formal power series $\xi = \sum_Q \xi_Q x^Q$, we let $\overline{\xi} = \sum_Q \|\xi_Q\| y^Q$. We also consider the series $\overline{\overline{\xi}}$, on one complex variable z, obtained as $\overline{\overline{\xi}} = \sum_Q \|\xi_Q\| z^{\|Q\|}$. Equivalently, $\overline{\overline{\xi}}(z) = \overline{\xi}(z,z)$. Given another series $\varpi = \sum_Q \varpi_Q y^Q$, we say that $\varpi \prec \xi$ if and only if $\|\varpi_Q\| \leq \|\xi_Q\|$ for all Q.

The proof of convergence of the formal diffeomorphism H, under assumptions that will be presented below, is based on the following result:

Theorem 2.2 (Cauchy Majorant Method) *For $i = 1, 2$, let φ_i be two holomorphic functions and ζ_i be two formal series as in (2.4) and (2.5). Suppose there exists $\delta > 0$ such that*

$$\delta\overline{\zeta_i} \prec \overline{\varphi}_i(y_1 + \overline{\zeta}_1, y_2 + \overline{\zeta}_2) .$$

Then the series ζ_i $(i = 1, 2)$ converge, hence defining a holomorphic change of coordinates.

Proof. From $\delta\overline{\zeta_i} \prec \overline{\varphi}_i(y_1 + \overline{\zeta}_1, y_2 + \overline{\zeta}_2)$ we obtain directly that

$$\overline{\overline{\zeta}}_1 + \overline{\overline{\zeta}}_2 \prec \frac{1}{\delta}(\overline{\varphi}_1(z + \overline{\overline{\zeta}}_1 + \overline{\overline{\zeta}}_2, z + \overline{\overline{\zeta}}_1 + \overline{\overline{\zeta}}_2) + \overline{\varphi}_2(z + \overline{\overline{\zeta}}_1 + \overline{\overline{\zeta}}_2, z + \overline{\overline{\zeta}}_1 + \overline{\overline{\zeta}}_2)) . \qquad (2.9)$$

Since the series φ_i are convergent by assumption, there exist a_0, $a > 0$ such that:

$$\frac{1}{\delta}(\overline{\overline{\varphi}}_1 + \overline{\overline{\varphi}}_2) \prec \frac{a_0 z^2}{1 - az} . \qquad (2.10)$$

From (2.9) and (2.10) we obtain

$$\begin{aligned}
\frac{1}{z}\left(\overline{\overline{\zeta}}_1+\overline{\overline{\zeta}}_2\right) &\prec \frac{1}{z\delta}\left(\overline{\overline{\varphi}}_1(z+\overline{\overline{\zeta}}_1+\overline{\overline{\zeta}}_2)+\overline{\overline{\varphi}}_2(z+\overline{\overline{\zeta}}_1+\overline{\overline{\zeta}}_2)\right)\\
&\prec \frac{1}{z}\left(\frac{a_0(z+\overline{\overline{\zeta}}_1+\overline{\overline{\zeta}}_2)^2}{1-a(z+\overline{\overline{\zeta}}_1+\overline{\overline{\zeta}}_2)}\right)\\
&\prec \frac{a_0 z[\frac{1}{z}(z+\overline{\overline{\zeta}}_1+\overline{\overline{\zeta}}_2)]^2}{1-az(\frac{1}{z}(z+\overline{\overline{\zeta}}_1+\overline{\overline{\zeta}}_2))}\ ,
\end{aligned}$$

Denoting by $u=\sum_i u_i z^i$ the series $\frac{1}{z}(\overline{\overline{\zeta}}_1+\overline{\overline{\zeta}}_2)$ we have just proved that

$$u \prec \frac{a_0 z(1+u)^2}{1-az(1+u)}\ . \tag{2.11}$$

We shall now compare u with the series $v=\sum_i v_i z^i$, that is a solution of

$$v=\frac{a_0 z(1+v)^2}{1-az(1+v)}\ .$$

or, equivalently, a solution of

$$\left(\sum v_i z^i\right)^2(-az-a_0z)+\left(\sum v_i z^i\right)(-az-2a_0z+1)-a_0z=0\ ,$$

The series v is convergent and, in particular, $v_1=a_0$. Notice that for every i, v_i is a polynomial with positive coefficients in the variables $v_1,\dots,v_{i-1}$, denoted by $P_i(v_1,\dots,v_{i-1})$. We may choose $a_0>u_1$ and it follows from (2.11) that

$$u_i\le P_i(u_1,\dots,u_{i-1})\ .$$

Now we show that $u\prec v$ by induction. Suppose that $u_j\le v_j$ for $j\le i-1$. The fact that P_i has positive coefficients for every i implies that

$$u_i\le P_i(u_1,\dots,u_{i-1})\le P_i(v_1,\dots,v_{i-1})=v_i\ .$$

Hence u is convergent, and consequently so is are ζ_i $(i=1,2)$. ■

We are now able to prove the convergence of H under the conditions below:

Theorem 2.3 (Poincaré Linearization Theorem) *Consider a system of differential equations as in (2.3). Assume that $\lambda_1\lambda_2\neq 0$ and that $\lambda_1/\lambda_2\notin\mathbb{R}_-$. Suppose also that neither λ_1/λ_2 nor λ_2/λ_1 belongs to $\mathbb{N}$. Then there is an analytic change of coordinates in which the system becomes linear.*

Proof. We first note that neither $\delta_{1,Q}$ nor $\delta_{2,Q}$ vanish independently of Q. Indeed if $\delta_{1,Q}$ vanishes then

$$\frac{\lambda_1}{\lambda_2}=\frac{q_2}{(1-q_1)}$$

So that whenever $q_2 = 0$, $\lambda_1 = 0$; when $q_1 = 0$, $\lambda_1/\lambda_2 \in \mathbb{N}$; and if $q_1 > 1$ then $\lambda_1/\lambda_2 \in \mathbb{R}_-$. A similar argument holds for $\delta_{2,Q}$. Therefore, $\zeta_{i,Q}$ can always be found in such a way that $\psi_{i,Q} = 0$. This means that there exists a formal change of coordinates that linearizes system (2.3). Now it must be shown that this change of coordinates is indeed convergent.

In fact our assumptions on λ_1 and λ_2 imply that there exists $\delta > 0$ such that $\inf_Q\{|\delta_{1,Q}|, |\delta_{2,Q}|\} \geq \delta$. This is a simple consequence of the convex separability theorem. From equations (2.7) and (2.8), we obtain for $j = 1, 2$

$$\begin{aligned} \delta\overline{\zeta}_j &\prec \sum_Q \delta_{j,Q}\|\zeta_{j,Q}\|y^Q + \overline{\psi}_j \\ &\prec \overline{\varphi}_j(y_1 + \overline{\zeta}_1, y_2 + \overline{\zeta}_2) + \sum_{k=1,2} \frac{\partial\overline{\zeta}_j}{\partial y_k}\overline{\psi}_k . \end{aligned}$$

Since $\overline{\psi}_k = 0$, we have

$$\delta\overline{\zeta_j} \prec \overline{\varphi}_j(y_1 + \overline{\zeta}_1, y_2 + \overline{\zeta}_2) .$$

Finally, Theorem 2.2 implies that this change of coordinates is convergent. ■

Assume now that $\lambda_1/\lambda_2 \in \mathbb{N}$. Although not necessarily linearizable the vector field has a very simple normal form in that case.

Lemma 2.1 *Suppose that $\lambda_1/\lambda_2 \in \mathbb{N}$. Then there is an analytic change of coordinates where the original system is given (in terms of vector fields) by*

$$X = (\lambda_1 y_1 + a y_2^n)\partial/\partial y_1 + \lambda_2 y_2 \partial/\partial y_2 .$$

where n is such that $\lambda_1 = n\lambda_2$ and $a \in \mathbb{C}$.

Proof. We first note that, under the assumptions above, $\inf_Q\{\delta_{1,Q}, \delta_{2,Q}\} \geq 1$ for all Q with exception to $Q = (0, n)$. In fact, the vector field hs a unique resonant relation ($\lambda_1 = n\lambda_2$ for some $n \in \mathbb{N}$). This implies that the coefficient of $\zeta_{1,(0,n)}$ in Equation (2.7) (which is given by $\delta_{1,(0,n)} = n\lambda_2 - \lambda_1$) vanishes. Since no more resonance relations appear, the vector field associated to the differential equation is then formally conjugated by a diffeomorphism H to

$$X = (n\lambda_2 y_1 + a y_2^n)\partial/\partial y_1 + \lambda_2 y_2 \partial/\partial y_2$$

for some $a \in \mathbb{C}$. The fact that $\inf_Q\{|\delta_{1,Q}|, |\delta_{2,Q}|\} \geq 1$ for all Q distinct from $(0, n)$ ensures that we can follow the proof of Theorem 2.3 in order to prove the convergence of H. ■

Now we shall obtain a characterization of singular holomorphic foliations in the case where $\lambda_1/\lambda_2 \in \mathbb{R}_-$.

Lemma 2.2 *If $\lambda_1/\lambda_2 \in \mathbb{R}_-$, then there is an analytic change of coordinates where the vector field associated to the original system is given by*

$$X = \lambda_1 y_1[1 + (h.o.t.)]\partial/\partial y_1 + \lambda_2 y_2[1 + (h.o.t.)]\partial/\partial y_2 .$$

In particular such vector field has two smooth transverse separatrizes.

Proof. It will be shown that there exists a convergent change of coordinates which allows us to suppose that φ_1 is divisible by the first variable and that φ_2 is divisible by the second one, where φ_1 and φ_2 are as in (2.3).

As before, we consider the change of coordinates $x_1 = y_1 + \zeta_1(y_1, y_2)$ and $x_2 = y_2 + \zeta_2(y_1, y_2)$ but setting:

$$\left\{ \begin{array}{ll} \psi_{1,Q} = 0 & \text{when } q_1 = 0 \text{ or } q_2 = 0 \\ \zeta_{1,Q} = 0 & \text{when } q_1 \neq 0 \text{ and } q_2 \neq 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll} \psi_{2,Q} = 0 & \text{when } q_1 = 0 \text{ or } q_2 = 0 \\ \zeta_{2,Q} = 0 & \text{when } q_1 \neq 0 \text{ and } q_2 \neq 0 \end{array} \right.$$

If this change of coordinates is indeed convergent then, in these appropriate coordinates, $\{y_1 = 0\}$ and $\{y_2 = 0\}$ are invariant which is the contents of the lemma. So we must analyze the expressions of $\delta_{1,Q}$ only in the case where $q_1 = 0$ or $q_2 = 0$. After all, when $q_1 \neq 0$ and $q_2 \neq 0$, $\zeta_{1,Q} = 0$, so that these terms do not count in the series $\overline{\zeta}_1$. Suppose that $q_1 = 0$. In this situation, we have

$$\begin{array}{rcl} \delta_{1,Q} & = & \lambda_2 q_2 - \lambda_1 \\ \Rightarrow \dfrac{\delta_{1,Q}}{\lambda_2} & = & q_2 - \dfrac{\lambda_1}{\lambda_2} > \varepsilon_1 \end{array}$$

for some $\varepsilon_1 > 0$. This guarantees that $|\delta_{1,Q}|$ is bounded from below by a constant $c > 0$. A similar argument implies that $|\delta_{2,Q}|$ is bounded from below either by a positive constant. More precisely, $|\delta_{2,Q}| > c > 0$ for all Q as above with $\|Q\| \geq 2$, i.e. such that $q_2 > 1$. The case $q_2 = 0$ is analogous.

Therefore, there exists $\delta > 0$ such that $\inf_Q\{|\delta_{1,Q}|, |\delta_{2,Q}|\} \geq \delta$, where the inf is taken between Q such that $q_1 = 0$ or $q_2 = 0$. Using the above mentioned fact along with Equations (2.7) and (2.8), we have:

$$\delta\overline{\zeta}_1 \prec \sum_Q \delta_{1,Q}\|\zeta_{1,Q}\|y^Q \prec \overline{\varphi}_1(y_1 + \overline{\zeta}_1, y_2 + \overline{\zeta}_2) + \frac{\partial\overline{\zeta}_1}{\partial y_1}\overline{\psi}_1 + \frac{\partial\overline{\zeta}_1}{\partial y_2}\overline{\psi}_2 \, .$$

By construction, the non-zero coefficients in $\overline{\zeta}_1$ are related only to monomials that are powers of y_1 or powers of y_2, i.e. there are no monomials that mix the two variables y_1 and y_2. On the other hand, all the non-zero terms that enter $\overline{\psi}_1$ and $\overline{\psi}_2$ are such that $q_1 = 0$ and $q_2 = 0$. So that the following stronger estimate holds:

$$\delta\overline{\zeta}_1 \prec \overline{\varphi}_1(y_1 + \overline{\zeta}_1, y_2 + \overline{\zeta}_2) \, .$$

A similar argument implies

$$\delta\overline{\zeta}_2 \prec \overline{\varphi}_2(y_1 + \overline{\zeta}_1, y_2 + \overline{\zeta}_2)$$

and the application of Theorem 2.2 guarantees the convergence of the series ■

2.2.2 Elementary Aspects of Saddle-Node Singularities

Dulac Normal Form and Consequences

We shall now consider the case of *saddle-nodes*, i.e. the case where $\lambda_1 \neq 0$ and $\lambda_2 = 0$. Obviously we can assume without loss of generality that $\lambda_1 = 1$ and

$\lambda_2 = 0$. We will see by successive appropriate changes of coordinates that, in this case, the system of ODEs (or the 1-form) that induces the foliation has a canonic representation. This is the contents of the following result.

Theorem 2.4 (Dulac) *Let $\mathcal{F}$ be a foliation which defines a saddle-node at $(0,0)$ in $\mathbb{C}^2$. Then, in appropriate coordinates (y_1, y_2), $\mathcal{F}$ is given by the holomorphic 1-form*

$$\omega = [y_1(1 + \lambda y_2^p) + y_2 R(y_1, y_2)]\, dy_2 \; - \; y_2^{p+1}\, dy_1 \, .$$

for some $p \in \mathbb{N}$.

Proof. Assume that

$$\begin{cases} \dot{x_1} = x_1 + \varphi_1(x_1, x_2) \\ \dot{x_2} = \varphi_2(x_1, x_2) \end{cases} , \tag{2.12}$$

where φ_1, φ_2 have order at least 2, is a representative of $\mathcal{F}$. First of all we will prove that there exists a change of coordinates $x_1 = y_1 + \zeta_1(y_1, y_2)$, $x_2 = y_2 + \zeta_2(y_1, y_2)$ in which the vector field associated to the differential equation above can be written in the form:

$$[y_1 + y_2 R(y_1, y_2)]\partial/\partial y_1 + y_2 \phi(y_1, y_2)\partial/\partial y_2, \, .$$

where $R(0,0) = \phi(0,0) = 0$.

Let us first note that, in that case, $\delta_{1,Q} = q_1 - 1$ whereas $\delta_{2,Q} = q_1$. We have that $\delta_{1,Q} = 0$ if and only if $q_1 = 1$ while $\delta_{2,Q} = 0$ if and only if $q_1 = 0$. This implies that we can always solve the correspondent to Equation (2.7) (resp. (2.8)) in order to $\zeta_{1,Q}$ (resp. $\zeta_{2,Q}$) for $q_1 \neq 1$ (resp. $q_1 \neq 0$). So, let us set

$$\left\{ \begin{array}{lll} \psi_{1,Q} = 0 & \text{whenever} & q_2 = 0 \\ \zeta_{1,Q} = 0 & \text{whenever} & q_2 \neq 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{lll} \psi_{2,Q} = 0 & \text{whenever} & q_2 = 0 \\ \zeta_{2,Q} = 0 & \text{whenever} & q_2 \neq 0 \end{array} \right.$$

for each index $Q = (q_1, q_2)$. Since $\|Q\| \geq 2$, it follows that $\delta_{1,Q} = q_1 - 1 \neq 0$ (resp. $\delta_{2,Q} = q_1 \neq 0$) whenever $\zeta_{1,Q} \neq 0$ (resp. $\zeta_{2,Q} \neq 0$). This proves that the two vector fields are formally conjugate. In order to prove the convergence of the change of coordinates we shall also note that $\delta_{1,Q} = q_1 - 1 \geq 1$ (resp. $\delta_{2,Q} = q_1 \geq 2$) whenever $\zeta_{1,Q} \neq 0$ (resp. $\zeta_{2,Q} \neq 0$). Thus we have

$$\delta\overline{\zeta}_1 \prec \overline{\varphi}_1(y_1 + \overline{\zeta}_1, y_2 + \overline{\zeta}_2) + \frac{\partial \overline{\zeta}_1}{\partial y_1}\overline{\psi}_1 + \frac{\partial \overline{\zeta}_1}{\partial y_2}\overline{\psi}_2$$

for $0 < \delta < 1$.

Notice that $\zeta_{1,Q} = 0$ whenever $q_2 \neq 0$ then $\overline{\zeta}_1$ depends only on y_1, so that $\frac{\partial \overline{\zeta}_1}{\partial y_2} = 0$. In particular, the non-zero coefficients of the series $\overline{\zeta}_1$ are such that $q_2 = 0$, then it follows from the above change of coordinates that all the monomials entering $\frac{\partial \overline{\zeta}_1}{\partial y_1}\overline{\psi}_1$ depend on y_2. Therefore these monomials do not appear in the series of ζ_1 and we conclude the stronger estimate

$$\delta\overline{\zeta}_1 \prec \overline{\varphi}_1(y_1 + \overline{\zeta}_1, y_2 + \overline{\zeta}_2) \, .$$

A similar argument implies that

$$\delta\overline{\zeta}_2 \prec \overline{\varphi}_2(y_1 + \overline{\zeta}_1, y_2 + \overline{\zeta}_2) \, .$$

The convergence of the coordinate change follows by applying the Cauchy Majorant Method. This allows us to suppose that φ_1 and φ_2 are divisible by x_2. In other words, the original system of ODEs is given by:

$$X(x_1, x_2) = [x_1 + x_2 R(x_1, x_2)] \frac{\partial}{\partial x_1} + x_2 \phi(x_1, x_2) \frac{\partial}{\partial x_2}$$

as we intended to prove.

Now, set $A(x_1, x_2) = x_1 + x_2 R(x_1, x_2)$ and $B(x_1, x_2) = x_2 \phi(x_1, x_2)$. The set $\{A = 0\} \cap \{B = 0\}$ is reduced to the origin $(0, 0)$. The ideal associated to the point $(0, 0)$ is therefore maximal and generated by x_1 and x_2. It follows from the appropriate version of Hilbert's Nullstellensatz that this maximal ideal is the radical of the ideal generated by A and B. In particular, there is $p + 1 \geq 2$ such that x_2^{p+1} belongs to the ideal generated by A and B (since x_2 itself cannot belong to this ideal). Next we expand both A and B in terms of x_2, i.e. we set $A = a_0(x_1) + \sum_{i=1}^{\infty} a_i(x_1) x_2^i$ and $B = \sum_{i=1}^{\infty} b_i(x_1) x_2^i$. The division of B by A in the ring $C\{x_1\}$ then gives us

$$B = AQ + x_2^{p+1} U(x_2)$$

being $Q = 0$ when $\{x_2 = 0\}$.

Indeed, $a_0'(x_1) = 1$ so that the rest does not depend on x_1. Besides $U(0) \neq 0$ since $p + 1$ is the smallest positive power of x_2 belonging in the ideal generated by A and B. Now consider the vector field

$$Y = \frac{1}{U} \frac{\partial}{\partial x_1} - \frac{Q}{U} \frac{\partial}{\partial x_2} \, ,$$

which satisfies $Y(0, 0) \neq (0, 0)$ since $U(0, 0) = U(0) \neq 0$. It follows from the fact that $Q = 0$ when $\{x_2 = 0\}$) that $\{x_2 = 0\}$ is a solution of Y. By the Flow Box Theorem, there exist coordinates (z_1, z_2) such that the vector field becomes

$$Y = \frac{\partial}{\partial z_1} \quad \text{and} \quad z_2 = x_2 \, ,$$

and the result follows. ■

Consider a foliation $\mathcal{F}$ as in the previous theorem, i.e. defining a saddle-node singularity. Hence, there exist appropriate coordinates where the associated vector field is given by the normal form:

$$X = [y_1(1 + \lambda y_2^p) + y_2 R(y_1, y_2)] \frac{\partial}{\partial y_1} + y_2^{p+1} \frac{\partial}{\partial y_2} \, .$$

In particular we can see that $\mathcal{F}$ admits a separatrix through the origin which is given, in the coordinates above, by $\{y_2 = 0\}$. Consider a loop on this separatrix encircling

the saddle-node singularity and let Σ be a transverse section to the separatrix passing through a point of the loop. We shall now compute the holonomy $h(z)$, where z is a local coordinate on Σ. This simple calculation will prove to be quite useful in the next sections. For the time being, it already provides a geometric interpretation of the number $p+1$ appearing in the normal form above.

Lemma 2.3 *The holonomy associated to the separatrix $\{y_2 = 0\}$ is given by $h(z) = z + z^{p+1} + \cdots$.*

Proof. In order to prove this result we proceed as follows. Set $y_1(t) = re^{2\pi it}$. Thus,

$$\begin{aligned}
\frac{dy_2}{dt} &= \frac{dy_2}{dy_1}\frac{dy_1}{dt} \\
&= \frac{y_2^{p+1}}{y_1(1+\lambda y_2^p) + y_2 R(y_1, y_2)} 2\pi i r e^{2\pi it} \\
&= \frac{y_2^{p+1}}{re^{2\pi it}[1+\lambda y_2^p + y_2 Q(re^{2\pi it}, y_2)]} 2\pi i r e^{2\pi it} \\
&= 2\pi i y_2^{p+1}(t)(1 + \text{h.o.t.})
\end{aligned} \tag{2.13}$$

where Q is holomorphic relatively to y_2. Denote $y_2(t) = \sum_{k\geq 1} a_k(t) z^k$ and with initial data $y_2(0) = z$, so that

$$\frac{dy_2}{dt} = \sum_{k\geq 1} a_k'(t) z^k \,. \tag{2.14}$$

By comparing the expressions obtained for dy_2/dt on (2.13) and on (2.14) and taking account that

$$y_2^{p+1}(t) = \left(\sum_{k\geq 1} a_k(t) z^k \right)^{p+1} \tag{2.15}$$

we see that $a_k'(t) = 0$ for $k \leq p$, i.e. the functions $a_k(t)$ are all constants for $k \leq p$. Since we have set $y_2(0) = z$, we have that $a_1(0) = 1$, $a_2(0) = \cdots = a_p(0) = 0$.

Now we compare the term $k = p+1$. Using (2.15) we obtain:

$$a_{p+1}'(t) = 2\pi i a_1^{p+1}(t) \,,$$

So that $a_{p+1}(t) = 2\pi it$ since $a_1(t) = 1$ for all t.

Since $h(z) = y_2(1)$ we conclude that $h(z) = z + 2\pi i z^{p+1} + \cdots$. By performing a change of coordinates we obtain the desired result, i.e. the holonomy can be written in the form $h(z) = z + z^{p+1} + \cdots$. □

Fatou Coordinates and the Leau Flower

In view of Lemma 2.3, we are naturally led to investigate the topological dynamics of diffeomorphisms which are tangent to the identity. In particular, we would like to understand the role of the multiplicity "$p+1$" on the topological dynamics of these diffeomorphisms. Here we follow closely the approach given in [Car-G].

So let us first analyze applications of the form $f(z) = z + z^{p+1} + \cdots$, in the prototypical case where $p = 1$. To begin with, let us apply a holomorphic change of coordinates, $A_1(z) = -1/z$, taking 0 to ∞. In these new coordinates f becomes

$$g(z) = z + 1 + b/z + \cdots$$

Fix $c \in \mathbb{R}^+$ sufficiently large so that $|g(z) - (z+1)| < 1/2$ for all $z \in \mathbb{C}$ such that $|z| > c$ and let $R_c = \{z \in \mathbb{C} : \operatorname{Re}(z) > c\}$. We will first prove that g is analytically conjugate to $z \mapsto z+1$ on R_c.

First of all we note that $g(R_1) \subset R_1$. In fact, since $\operatorname{Re}(z) > -|z|$ we have

$$\frac{1}{2} < 1 + \operatorname{Re}\left(\frac{b}{z} + O\left(\frac{1}{z^2}\right)\right) < \frac{3}{2} \tag{2.16}$$

and therefore

$$\operatorname{Re}(g(z)) = \operatorname{Re}(z) + 1 + \operatorname{Re}\left(\frac{b}{z} + O\left(\frac{1}{z^2}\right)\right) > \operatorname{Re}(z) > c\,.$$

This implies that the map $\varphi_n(z) = g^n(z) - n - b\log n$ is well defined on R_c, where $g^n = g \circ \cdots \circ g$, n times. If φ_n converges to a holomorphic function φ then g is holomorphically conjugate to the translation $T(z) = z+1$. Indeed, one has

$$\lim_{n\to\infty} \varphi_n(g(z)) = \lim_{n\to\infty} [\varphi_{n+1}(z) + 1 + b\log(1+1/n)] = \lim_{n\to\infty} \varphi_{n+1}(z) + 1\,.$$

The convergence of φ_n is the contents of the next lemma.

Lemma 2.4 *The sequence φ_n converges to a conformal function φ.*

Proof. We note that $g^n(z) = z + n + O(1/n)$. The estimate (2.16) implies that

$$\frac{n}{2} \leq |g^n(z)| \leq |z| + 2n\,.$$

This can be easily verified by induction on n. In fact for the upper bound estimate we have

$$|g(z)| \leq |z| + |1 + O(1/z)| \leq |z| + 2\,.$$

Assuming that the estimate is valid for n then

$$|g^{n+1}(z)| = |g^n(g(z))| \leq |g(z)| + 2n \leq |z| + 2(n+1)$$

Relatively to the lower estimate we have that

$$|g^n(z)| > \mathrm{Re}(g^n(z)) > \mathrm{Re}(z) + \frac{n}{2} > \frac{n}{2}\,.$$

We have therefore that $g^{k+1}(z) = g^k(z) + 1 + \frac{b}{g^k(z)} + O(1/k^2)$. Thus we obtain that

$$\varphi_{k+1} - \varphi_k(z) = b[\log k - \log(k+1)] + \frac{b}{g^k(z)} + O(1/k^2) = O(1/k)\,.$$

Hence, for $z \in R_1$ the following estimate holds

$$|\varphi_n(z) - z| \leq |\varphi_1(z) - z| + \sum_{k=1}^{n-1} |\varphi_{k+1}(z) - \varphi_k(z)| = O(\log n)\,.$$

It remains to prove that φ_n is uniformly convergent on the compact subsets of R_c. We have the estimates

$$\begin{aligned}
\varphi_{n+1}(z) - \varphi_n(z) &= b\log n - b\log(n+1) + g^{n+1}(z) - g^n(z) - 1\\
&= -\frac{b}{n} + \frac{b}{g^n(z)}\left(= O\left(\frac{1}{n^2}\right)\right)\\
&= b\left[\frac{1}{n + b\log n + \varphi_n(z)} - \frac{1}{n}\right] + O(\frac{1}{n^2})\\
&= \frac{1}{n^2}O\left(|b\log n + \varphi_n(z)|\right) + O\left(\frac{1}{n^2}\right)\\
&= O\left(\frac{\log n}{n^2}\right),
\end{aligned}$$

so that $\sum |\varphi_{n+1}(z) - \varphi_n(z)| < \infty$. Since all the φ_n are conformal, so is the uniform limit φ. ■

Next we note that we can extend φ analytically to any domain Ω, contained in the domain of g, verifying $g(\Omega) \subset \Omega$ and such that $\mathrm{Re}(g^n(z))$ tends to ∞, for $z \in \Omega$. In fact we can construct one such invariant domain Ω with smooth boundary, as is shown in Figure 2.1. We have therefore that $f(z) = z + z^2 + \cdots$ is conjugate to the translation $T(z) = z + 1$ on the cardioid-shaped region, $A_1^{-1}(\Omega)$, as shown in Figure 2.2. The set $A_1^{-1}(\Omega)$ is therefore what we call an attracting petal centered at an attracting direction. Basically this means that $A_1^{-1}(\Omega)$ is an invariant set whose orbit converges to the origin tangentially to the direction v.

Definition 2.2 *Let $f \in \mathrm{Diff}(\mathbb{C}, 0)$ be such that $f(z) = z + az^{p+1} + \cdots$, where $a \neq 0$. An attracting petal for f centered at an attracting direction v is a simply connected open set P such that*

a) $0 \in \partial P$

b) $f(P) \subseteq P$

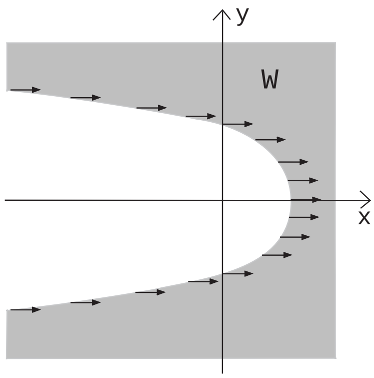

Figure 2.1: Invariant domain by g

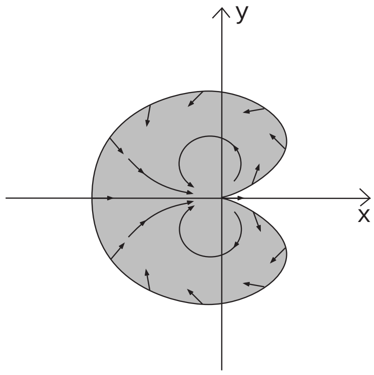

Figure 2.2: Cardioid-shaped Dynamics of f

c) $\lim_{n\to+\infty} f^{(n)} = 0$ *with* $\lim_{n\to+\infty} \frac{f^n(z)}{|f^n(z)|} = v$ *for all* $z \in P$

A repelling petal centered at a repelling direction v is an attracting petal for f^{-1} centered at the attracting direction v for f^{-1}.

We should note that an attracting (resp. repelling) direction is an element v of $\mathbb{C}$ such that $av^k/|a| \in \mathbb{R}_-$ (resp. $av^k/|a| \in \mathbb{R}_+$). Moreover, the union of the attracting petals and the repelling ones constitutes a neighborhood of the origin.

In the case that $p = 1$ we have exactly one attracting petal (resp. direction) and one repelling petal (resp. direction). The general case where $p \geq 2$ can be treated in a similar way. In fact, this case can be reduced to previous one. Let $f_p(z) = z + z^{p+1} + \cdots$. Up to conjugation by an homothety, we can suppose that $f_p(z) = z + \frac{1}{p}z^{p+1} + \cdots$. Now conjugating by $A_p(z) = -z^{1/p}$ we obtain

$$A_p^{-1} \circ f_p \circ A_p(z) = z\left(1 + \frac{1}{p}z + \cdots\right)^p = z + z^2 + \cdots .$$

Thus we are reduced to the case $p = 1$, which has been analyzed before. We note that the sectors $|\arg z - 2k\pi/p| < \pi/p$ are mapped conformally, by A_p^{-1}, onto the plane minus the negative real axis. This implies that in the case $p \geq 2$ we have essentially a ramification of order p of the previous case. The mapping f_p has therefore p *attracting petals* P_k. The petals are (invariant domains) bounded by piecewise analytic Jordan curves $(c_1, \ldots, c_k)$ and they are symmetric relatively to the rays $\arg z = (2k\pi)/p$. At the origin, c_i has two tangents $\arg z = (2i \pm 1)\pi/p$. The final picture is summarized by the theorem below.

Theorem 2.5 (Flower Theorem) *Let $f \in \mathrm{Diff}(\mathbb{C}, 0)$ be given by $f(z) = z + az^{p+1} + \cdots$, where $a \neq 0$. Denote by $v_1^+, \ldots, v_p^+$ the attracting directions and by $v_1^-, \ldots, v_p^-$ the repelling ones. Assume that they are ordered by the following rule: starting at v_i^+ and moving in the counterclockwise direction we first meet v_i^- and then v_{i+1}^+. Then*

a) For each v_i^+ (resp. v_i^-) there exists an attracting (resp. repelling) petal P_i^+ (resp. P_i^-) centered at v_i^+ (resp. v_i^-)

b) The union of all attracting and repelling petals constitutes a neighborhood of the origin

c) $P_i^+ \cup P_j^+ =$ and $P_i^- \cup P_j^- =$ for $i \neq j$

d) The diffeomorphism f is holomorphically conjugated to the translation map $T(z) = z + 1$ on each attracting petal.

We recall that a domain V is called a *Leau domain* if f is conjugate to the translation $T(z) = z + 1$ on V and also if the sequence f^n converges to a point on ∂V. The cardioid-shaped region illustrated on Figure 2.2 is an example of a Leau domain.

2.2.3 Some Normal Forms in Higher Dimensions

The Cauchy Majorant Method was essential to prove the convergence of the formal conjugating diffeomorphisms for generic vector fields in dimension 2. We begin this section noticing that the generalization of this result to higher dimension is straightforward. We will not prove it.

Theorem 2.6 (Cauchy Majorant Method in Several Variables) *Let ϕ_i, $i = 1, \ldots, n$, be n holomorphic functions with trivial linear part and let ζ_i, $i = 1, \ldots, n$, be n formal series. Consider the change of coordinates*

$$x_i = y_i + \zeta_1(y_1, \ldots, y_n) \text{ where } \zeta_i(y_1, \ldots, y_n) = \sum_{\|Q\|>1} \zeta_{i,Q} y^Q \quad (i = 1, \ldots, n) \tag{2.17}$$

and assume the existence of $\delta > 0$ such that

$$\delta \overline{\zeta}_i \prec \overline{\varphi}_i(y_1 + \overline{\zeta}_1, \ldots, y_n + \overline{\zeta}_n)$$

for all $i = 1, \ldots, n$. Then the series of ζ_i converges and hence defines a holomorphic change of coordinates. □

We shall obtain the correspondent to Theorem 2.3 for several variables. Before that let us introduce some definitions.

Definition 2.3 *Suppose that the origin is a singular point of a vector field X on $(\mathbb{C}^n, 0)$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the vector of eigenvalues of $\mathrm{Jac}X(0)$. We say that the eigenvalues are resonant if, for some $i \in \{1, \ldots, n\}$, there exists $I = (i_1, \ldots, i_n) \in \mathbb{N}_0^n$ with $\sum_{j=1}^n i_j \geq 2$ such that*

$$\lambda_i = (I, \lambda) = i_1\lambda_1 + \ldots + i_n\lambda_n$$

i.e. if at least one of the eigenvalues can be written as a non-trivial positive linear combination of all of them. In this case the monomials $x^I \partial/\partial x_i$ are said to be resonant. Like in the two dimensional case, if $\dim\{m \in \mathbb{Z}^n : (m, \lambda) = 0\} = k$ then X is said to be k-resonant.

Definition 2.4 *We say that a n-tuple* $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ *belongs to the Poincaré's domain if the convex hull of the complex numbers* $\lambda_1, \ldots, \lambda_n$*, i.e. if the set* $\{z \in \mathbb{C} : t_1\lambda_1 + \cdots + t_n\lambda_n = z\, ,\, t_1 + \cdots + t_n = 1\}$*, does not contain the origin* $0 \in \mathbb{C}$*. Otherwise we say that* $(\lambda_1, \ldots, \lambda_n)$ *belongs to the Siegel domain.*

A linear vector field X *defined on* $\mathbb{C}^n$ *is said to be of Poincaré-type (resp. Siegel-type) if its spectrum is in the Poincaré's domain (resp. Siegel's domain).*

To belong to the Poincaré domain is equivalent to the existence of a straight line through the origin such that all the eigenvalues $\lambda_1, \ldots, \lambda_n$ belong to a same half-plane defined by this straight line.

Theorem 2.7 (Poincaré Linearization Theorem) *Let* X *be a holomorphic vector field defined in a neighborhood of the origin of* $\mathbb{C}^n$ *such that its linear part is of Poincaré-type. Let* $\lambda_1, \ldots, \lambda_n$ *be the eigenvalues of* $\mathrm{Jac}X(0)$*. Assume that there there exists no resonance relation. Then there exists a holomorphic change of coordinates that linearizes* X*.*

Proof. The idea of the proof is the same as in the case of dimension 2. Substituting Equations (2.17) in the differential equation associated to X, we obtain the following relations

$$\sum_{\|Q\|>1} (\delta_{i,Q}\zeta_{i,Q} + \psi_{i,Q})y^Q = \varphi_i(y_1 + \zeta_1, \ldots, y_n + \zeta_n) - \sum_{k=1}^{n} \frac{\partial \zeta_i}{\partial y_k}\psi_k\ ,$$

for $i = 1, \ldots, n$, where $y^Q = y_1^{q_1} \ldots y_n^{q_n}$ and $\delta_{i,Q} = q_1\lambda_1 + \cdots + q_n\lambda_n - \lambda_i$, with $q_i \in \mathbb{N}$. Due to the non-resonant assumption, we have that $\delta_{i,Q} \neq 0$ for all $i = 1, \ldots, n$. Thus the equations above are solvable in order to $\zeta_{i,Q}$, for all i and Q. and, consequently, X is formally linearizable. In order to prove the convergent it is sufficient to prove that $\{|\delta_{i,Q}|\}_{i,Q}$ is bounded from below by a positive constant. The proof goes as in the two dimensional case.

Let δ be a positive constant such that $\inf_{i,Q}\{\delta_{i,Q}|\} \geq \delta > 0$. Therefore,

$$\begin{aligned} \delta\overline{\zeta}_j &\prec \sum_{Q} \delta_{j,Q}\|\zeta_{j,Q}\|y^Q + \overline{\psi}_j \\ &\prec \overline{\varphi}_j(y_1 + \overline{\zeta}_1, \ldots, y_n + \overline{\zeta}_n) + \sum_{k=1}^{n} \frac{\partial \overline{\zeta}_j}{\partial y_k}\overline{\psi}_k\, . \end{aligned}$$

Since $\overline{\psi}_k = 0$, we have

$$\delta\overline{\zeta_j} \prec \overline{\varphi}_j(y_1 + \overline{\zeta}_1, \ldots, y_n + \overline{\zeta}_n)\ .$$

The convergence of the desired coordinate change results by applying Theorem 2.2. ■

Now we shall approach the case of a saddle-node singularity in dimension 3. First we will consider the case where only one of the eigenvalues is *zero*.

Theorem 2.8 *Let X be a holomorphic vector field defined in a neighborhood of the origin of $\mathbb{C}^3$. Denote by $\lambda_1, \lambda_2, \lambda_3$ the eigenvalues of the linear part of X at the origin. Assume that $\lambda_3 = 0$. Assume also that λ_1, λ_2 are non-vanishing eigenvalues not satisfying any resonance relation and belonging to the Poincaré domain. Then there is an analytic change of coordinates where the original system is given (in terms of vector fields) by*

$$X = [\lambda_1 y_1 + y_3 \psi_1(y_1, y_2, y_3)]\partial/\partial y_1 + [\lambda_2 y_2 + y_3 \psi_2(y_1, y_2, y_3)]\partial/\partial y_2 + y_3 H(y_1, y_2, y_3)\partial/\partial y_3 \,.$$

with $\psi_1(0,0,0) = \psi_2(0,0,0) = H(0,0,0) = 0$. In particular, $\{y_3 = 0\}$ is an invariant 2-plane.

Proof. Using the same notation as before, we consider a formal change of coordinates such that:

$$\left\{ \begin{array}{lll} \psi_{1,Q} = 0 & \text{when} & q_3 = 0 \\ \zeta_{1,Q} = 0 & \text{when} & q_3 \neq 0 \end{array} \right. ; \left\{ \begin{array}{lll} \psi_{2,Q} = 0 & \text{when} & q_3 = 0 \\ \zeta_{2,Q} = 0 & \text{when} & q_3 \neq 0 \end{array} \right. \text{ and } \left\{ \begin{array}{lll} \psi_{3,Q} = 0 & \text{when} & q_3 = 0 \\ \zeta_{3,Q} = 0 & \text{when} & q_3 \neq 0 \end{array} \right.$$

As in Theorem 2.7, the assumption over the eigenvalues guarantee the existence of a positive constant $\delta > 0$ such that $\inf_i\{|\delta_{i,Q}| : q_1 q_2 q_3 \neq 0\} > \delta > 0$. Hence,

$$\delta\overline{\zeta}_i \prec \overline{\varphi}_j(y + \overline{\zeta}) + \frac{\partial\overline{\zeta}_i}{\partial y_1}\overline{\psi}_1 + \frac{\partial\overline{\zeta}_i}{\partial y_2}\overline{\psi}_2 + \frac{\partial\overline{\zeta}_i}{\partial y_3}\overline{\psi}_3 \,.$$

Notice that $\overline{\zeta}_i$ depends only on y_1 and y_2, so that the last term in the above estimate vanishes for $i = 1, 2, 3$. On the other hand, by construction, all the non-zero coefficients $\psi_{i,Q}$ are associated to monomials that depend on y_3, in such a way that we are able to conclude:

$$\delta\overline{\zeta_i} \prec \overline{\varphi}_i(y + \overline{\zeta}) \,,$$

and, as before, the application of Theorem 2.2 concludes the proof. ■

Next, let us consider the case where two eigenvalues are equal to *zero*. We will notice that the method used in the previous case to prove the existence of an invariant plane does not work in the present situation.

Let X be a holomorphic vector field defined in a neighborhood of the origin of $\mathbb{C}^3$, whose linear part has eigenvalues $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 \neq 0$. We could try to see if the previous method allows us to prove the existence of an invariant plane. Let X be given, in local coordinates, by:

$$\varphi_1(x_1, x_2, x_3)\partial/\partial x_1 + \varphi_2(x_1, x_2, x_3)\partial/\partial x_2 + [\lambda_3 x_3 + \varphi_1(x_1, x_2, x_3)]\partial/\partial x_3$$

where φ_1, φ_2, φ_3 are holomorphic functions of order at least 2. Using the same method it is possible to prove the existence of a formal change of coordinates such that φ_1 is divisible by x_1, and φ_2, φ_3 are divisible by x_2. In this case, the plane $\{x_1 = 0\}$ is invariant under X. However, the method used in the above situation does not allow us to prove the convergence of the series.

Let us now go back to the saddle-node case with exactly one eigenvalue equal to zero.

Theorem 2.9 *Let X be a holomorphic vector field defined in a neighborhood of the origin of $\mathbb{C}^3$, with eigenvalues $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_3 = 0$. Let $n_0 \in \mathbb{N}$ be an arbitrary positive integer. Then there is an analytic change of coordinates where the original system is given (in terms of vector fields) by*

$$X = F(y_1, y_2, y_3)\partial/\partial y_1 + G(y_1, y_2, y_3)\partial/\partial y_2 + H(y_1, y_2, y_3)\partial/\partial y_3\,,$$

such that $F(0,0,y_3) = G(0,0,y_3) \equiv 0$, i.e, the axis $\{y_1 = y_2 = 0\}$ is invariant under X.

Proof. Consider a change of variables of the form

$$x_1 = y_1 + \sum_{i=2}^{N} a_i y_3^i\,, \qquad x_2 = y_2 + \sum_{i=2}^{N} b_i y_3^i\,, \qquad x_3 = y_3\,. \tag{2.18}$$

We have to show that N, a_i, b_i can be chosen so as to fulfill our requirements. In the coordinates (y_1, y_2, y_3) the vector field X is given by

$$\begin{bmatrix} 1 & 0 & -\sum_{i=2}^{N} i a_i y_3^{i-1} \\ 0 & 1 & -\sum_{i=2}^{N} i b_i y_3^{i-1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ G_1 \\ H_1 \end{bmatrix}. \tag{2.19}$$

In the above formula, the functions F_1, G_1, H_1 admit respectively the expressions below:

$$F_1 = \lambda_1 y_1 + \sum_{i=2}^{N} \lambda_1 a_i y_3^i + (\sum_{i=2}^{N} a_i y_3^i)(\sum_{i=2}^{N} b_i y_3^i)(f_{1,1}(y_3) + (*)) + f_{1,2}(y_3) + (**), \tag{2.20}$$

$$G_1 = \lambda_2 y_2 + \sum_{i=2}^{N} \lambda_2 b_i y_3^i + (\sum_{i=2}^{N} a_i y_3^i)(\sum_{i=2}^{N} b_i y_3^i)(g_{1,1}(y_3) + (***)) + g_{1,2}(y_3) + (****), \tag{2.21}$$

$$H_1 = h(y_3) + (*****). \tag{2.22}$$

In the above equations, the components represented as $(*), \ldots, (*****)$ do not contain neither constants nor terms depending only on y_3. In other words these components belong to the ideal $I(y_1) \cup I(y_2)$.

On the other hand, we want to consider the terms depending only on y_3 which appear in the 2 first coordinates of X. After performing the matricial product (2.29), these coordinates are respectively given by

$$\sum_{i=2}^{N} \lambda_1 a_i y_3^i + (\sum_{i=2}^{N} a_i y_3^i)(\sum_{i=2}^{N} b_i y_3^i) f_{1,1}(y_3) + f_{1,2}(y_3) - (\sum_{i=2}^{N} i a_i y_3^{i-1}) h(y_3) \text{ and} \tag{2.23}$$

$$\sum_{i=2}^{N} \lambda_2 b_i y_3^i + (\sum_{i=2}^{N} a_i y_3^i)(\sum_{i=2}^{N} b_i y_3^i) g_{1,1}(y_3) + g_{1,2}(y_3) - (\sum_{i=2}^{N} i b_i y_3^{i-1}) h(y_3)\,. \tag{2.24}$$

We now consider the Taylor expansions of $f_{1,1}, g_{1,1}$, namely we set

$$f_{1,1}(y_3) = \alpha_0^{(1)} + \alpha_1^{(1)} y_3 + \cdots \quad \text{and} \quad g_{1,1}(y_3) = \beta_0^{(1)} + \beta_1^{(1)} y_3 + \cdots . \tag{2.25}$$

Furthermore the assumption on the linear part of X and the expression of the change of coordinates in (2.18) allow us to write

$$\begin{aligned} f_{1,2}(y_3) = \alpha_2^{(2)} y_3^2 + \cdots \quad &; \quad g_{1,2}(y_3) = \beta_2^{(2)} y_3^2 + \cdots , \\ h(y_3) = c_k y_3^k + \cdots \quad & \quad\ (c_k \neq 0, \ k \geq 2) . \end{aligned} \tag{2.26}$$

Because we just want to cancel the coefficients of degree less than $n_0 + 1$ which depend solely on y_3, the Equations (2.23) and (2.24) can respectively be replaced by (2.27) and (2.28) without loss of generality, where

$$\begin{aligned} &\sum_{i=2}^{n_0} \lambda_1 a_i y_3^i + (\sum_{i=2}^{n_0-1} a_i y_3^i)(\sum_{i=2}^{n_0-1} b_i y_3^i)(\alpha_0^{(1)} + \alpha_1^{(1)} y_3 + \cdots + \alpha_{n_0-2}^{(1)} y_3^{n_0-2}) + \\ &+ (\alpha_2^{(2)} y_3^2 + \cdots + \alpha_{n_0}^{(2)} y_3^{n_0}) - (\sum_{i=2}^{n_0-k} i a_i y_3^{i-1})(c_k y_3^k + \cdots + c_{n_0} y_3^{n_0}) \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} &\sum_{i=2}^{n_0} \lambda_2 b_i y_3^i + (\sum_{i=2}^{n_0-1} a_i y_3^i)(\sum_{i=2}^{n_0-1} b_i y_3^i)(\beta_0^{(1)} + \beta_1^{(1)} y_3 + \cdots + \beta_{n_0-2}^{(1)} y_3^{n_0-2}) + \\ &+ (\beta_2^{(2)} y_3^2 + \cdots + \beta_{n_0}^{(2)} y_3^{n_0}) - (\sum_{i=2}^{n_0-k} i b_i y_3^{i-1})(c_k y_3^k + \cdots + c_{n_0} y_3^{n_0}) . \end{aligned} \tag{2.28}$$

In view of the two formulas (2.27) and (2.28) above, the proposition is reduced to the following claim.

Claim: The coefficients $\alpha_0^{(1)}, \beta_0^{(1)}$ are constant (that is they do not depend on a_i, b_i). Furthermore $\alpha_l^{(1)}, \beta_l^{(1)}$ are polynomials on $a_1, b_1, \ldots, a_l, b_l$ for $1 \leq l \leq n_0 - 2$. In addition, $\alpha_l^{(2)}, \beta_l^{(2)}$ are polynomials on $a_1, b_1, \ldots, a_{l-1}, b_{l-1}$ for $1 \leq l \leq n_0$. In particular none of these coefficients depend on a_{n_0}, b_{n_0}.

Proof of the Claim. The facts concerning $\alpha_l^{(1)}, \beta_l^{(1)}$ $(l = 0, 1, \ldots, n_0 - 2)$ are immediate consequences of (2.25) and of the form of the change of coordinates (2.18).

The assertion regarding $\alpha_l^{(2)}, \beta_l^{(2)}$ $(l = 1, \ldots, n_0)$ has a similar justificative which however deserves further comments.

When we perform the change of coordinates given by (2.18) and consider, for instance, dy_1/dT, we obtain

$$\lambda_1 y_1 + \sum_{i=2}^{N} \lambda_1 a_i y_3^i + (\sum_{i=2}^{N} a_i y_3^i)(\sum_{i=2}^{N} b_i y_3^i)(f_1^1(y_3) + (*)) + f_2^1(y_3) + (**) ,$$

for appropriate holomorphic functions f_1^1, f_2^1. Moreover $(*)$ represents terms which do not depend solely on the variable y_3 while $(**)$ represents terms which are neither constant nor depend solely on the variable y_3. Fix a monomial of f_2^1 having degree $r \in \mathbb{N}$. The coefficients entering into the term $\alpha_q^{(2)}$ are those corresponding to the monomials $y_1^{q_1} y_2^{q_2} y_3^{q_3}$ such that $1 < q_1 + q_2 + q_3 < r$. This implies the claim. The proof of the theorem is over. ■

2.3 Advanced Aspects of Singularities

2.3.1 Saddle-Node in Dimension 2

Basic Properties

As was seen earlier, for a foliation $\mathcal{F}$ (or vector field) with a saddle-node singularity in $(0,0) \in \mathbb{C}^2$ there is a holomorphic change of coordinates, by means of which $\mathcal{F}$ may be given by the 1-form (the Dulac's normal form):

$$\omega(y_1, y_2) = [y_1(1 + \lambda y_2^p) + y_2 R(y_1, y_2)]\, dy_2 \; - \; y_2^{p+1}\, dy_1 \,,$$

where $\lambda \in \mathbb{C}$, $p \in \mathbb{N}^*$ and the order of R at $(0,0)$ with respect to y_1 is at least $p+1$.

Let us consider the *formal* change of coordinates:

$$(y_1, y_2) \mapsto (\varphi(y_1, y_2), y_2) \,, \tag{2.29}$$

taking ω into its formal normal form $\omega_{p,\lambda}$, where $\varphi(y_1, y_2) = y_1 + \sum_{i=1}^{\infty} a_i(y_1) y_2^i$ and

$$\omega_{p,\lambda}(y_1, y_2) = [y_1(1 + \lambda y_2^p)]\, dy_2 \; - \; y_2^{p+1}\, dy_1 \,.$$

A careful look shows that the functions $a_1(y_1)$ are holomorphic on a common neighborhood of $0 \in \mathbb{C}$, though the change of variables is *not* necessarily convergent. In other words, the 1-forms, ω and $\omega_{p,\lambda}$ are *not* holomorphically conjugate, in general.

Example 2.1 *Let us consider the following example due to Euler:*

$$(y - x^2)dx - x^2 dy = 0 \,.$$

It admits a formal solution given by

$$y(x) = \sum_{n=1}^{\infty} n! x^{n+1} \,,$$

which does not converge in any neighborhood of 0*. Therefore the vector field cannot be holomorphically conjugated to its formal normal form.*

In this section we are going to discuss the problem of the analytic classification of saddle-nodes following [Ma-R]. Our purpose is to summarize the main points of [Ma-R] by explaining the role of the sectorial normalizations along with the analytic

invariants of the diffeomorphisms arising from "changing the sector". By a *sector V with vertex at* 0, we mean an angular sector of angle $\theta < 2\pi$, intersected with the ball $B_r \subset \mathbb{C}$ of radius r centered in $0 \in \mathbb{C}$.

Let f be a holomorphic function in $U \times V$, where U is a neighborhood of $0 \in \mathbb{C}$ and $V \subseteq \mathbb{C}$ is a sector with vertex at 0. We say that $\hat{f} = \sum_{i=1}^{\infty} a_i(y_1)y_2^i$ is the *asymptotic expansion* of f at $0 \in \mathbb{C}$ if for each $n \in \mathbb{N}$ there exists $A_n(y_1) > 0$ such that for every $y_1 \in U$:

$$\left| f(y_1, y_2) - \sum_{r=0}^{n-1} a_r(y_1)y_2^r \right| \leq A_n(y_1)y_2^n \, .$$

The idea is that certain formal series can be realized as asymptotic expansions of holomorphic functions that are defined on sectors of angles $\theta < 2\pi$ and sufficiently small radius r. In other words they are not defined on a neighborhood of zero, otherwise the series would have to converge since they would agree with the Taylor series of the functions.

The next theorem, due to H. Hukuara, T. Kimura and T. Matuda [H-K-M] implies that, although in general the 1-forms ω and $\omega_{p,\lambda}$ are not holomorphically conjugate in neighborhoods of $(0,0) \in \mathbb{C}^2$, they are actually holomorphically conjugate in conveniently chosen sectors.

Theorem 2.10 (Hukuara-Kimura-Matuda [H-K-M]) *Let φ be a formal series as defined in (2.29), i.e. the change of coordinates that takes ω into $\omega_{p,\lambda}$. Then, for every sector $V \subseteq \mathbb{C}$ of angle less than $\frac{2\pi}{p}$, there exists a bounded holomorphic map*

$$\begin{array}{rcl} \Phi_V : B_r \times (V \setminus \{0\}) & \to & \mathbb{C} \times (V \setminus \{0\}) \\ (y_1, y_2) & \mapsto & (\varphi_v(y_1, y_2), y_2) \, , \end{array}$$

such that:

1. *$\Phi_V^* \omega \wedge \omega_{p,\lambda} = 0$;*
2. *φ is the asymptotic expansion of Φ_V at $0 \in \mathbb{C}$.*

Φ_V is called a normalizing application.

To simplify, we will consider the particular case of $p = 1$. Note that the cases $p > 1$ may be dealt with in a similar way, except that the number of sectors which are necessary to cover the ball centered in $0 \in \mathbb{C}$ increases. The general procedure would be a straightforward generalization of the case $p = 1$ somehow in the spirit of the Flower Theorem (2.5) compared to the case $p = 1$ (cardioid-shaped region).

According to Hukuara-Kimura-Matuda Theorem, we can cover the neighborhood B_r of $0 \in \mathbb{C}$ with two sectors of angles less than 2π. To fix ideas, we may suppose that $V_1 = B_r \cap \{z \in \mathbb{C}; \arg \in [0, 5\pi/4) \cup (7\pi/4, 2\pi]\}$ and $V_2 = B_r \cap \{z \in \mathbb{C}; \arg \in [0, \pi/4) \cup (3\pi/4, 2\pi]\}$. Notice that $V_1 \cap V_2$ has two connected components. The bisectrix of one of them is the positive real semi-axis, therefore we shall denote it

by V^+. Analogously for V^-, the component whose bisectrix is the negative real semi-axis.

Let us consider the foliation induced by the equation $\omega_{1,\lambda} = 0$ on a neighborhood of $(0,0) \in \mathbb{C}^2$. Let $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) be the restriction of the foliation to $B_r \times V_1$ (resp. $B_r \times V_2$). Naturally $\mathcal{F}_1$ and $\mathcal{F}_2$ coincide in the intersection of the domains, so we denote the foliation in $B_r \times V^+$ (resp. $B_r \times V^-$) by $\mathcal{F}^+$ (resp. $\mathcal{F}^-$).

Theorem 2.10 assures the existence of normalizing applications H_1 and H_2 defined on $B_r \times (V_1 \setminus \{0\})$ and $B_r \times (V_2 \setminus \{0\})$, respectively. In the following, our aim is to understand the behavior of the applications that are responsible for the changing of sectors. First of all, the change of sectors $H_1 \circ H_2^{-1}$ gives rise to two diffeomorphisms denoted by g^+ and g^-. The first one corresponds to the restriction of $H_1 \circ H_2^{-1}$ to V^+ while the second one corresponds to the restriction of the same map to V^-. The next proposition gives a characterization for these functions.

Proposition 2.1 *The diffeomorphism $g^+ = H_1 \circ H_2^{-1}|_{V^+}$ is a translation and the diffeomorphism $g^- = H_1 \circ H_2^{-1}|_{V^-}$ is tangent to the identity.*

Before proving this proposition we will give a geometric approach to g^+ and g^-. Firstly, notice that the solutions of $\omega_{1,\lambda} = 0$, i.e. the leaves of the foliation, are given by

$$y_1(y_2) = c y_2^{\lambda} \exp\left(-\frac{1}{y_2}\right) , \tag{2.30}$$

with $c \in \mathbb{C}$. So each leaf of $\mathcal{F}^+$ (resp. $\mathcal{F}^-$) is in correspondence with $c \in \mathbb{C}$. In other words, the leaf space is isomorphic to $\mathbb{C}$, being parameterized by the constants $c \in \mathbb{C}$. Therefore g^+ (or g^-, depending on whether $Re(x) > 0$ or $Re(x) < 0$) is defined on $\mathbb{C}$ and $g^+(c)$ (or $g^-(c)$) is the corresponding leaf when we change sectors.

The following is classical after [Ma-R], we follow however the discussion in [R1]. Fix a sector V and let us consider the group of automorphisms $\Lambda_{\omega_{1,\lambda}}(V)$, such that each element is a diffeomorphism ϕ defined on V with the following properties:

1. $\phi(y_1, y_2) = (\varphi(y_1, y_2), y_2)$,
2. ϕ is asymptotic to the identity,
3. ϕ preserves the foliation $\mathcal{F}$ induced by $\omega_{1,\lambda} = 0$, i.e. $\phi^*\omega_{1,\lambda} \wedge \omega_{1,\lambda} = 0$.

Note that if H_1, H_2 are normalizing applications on $B_r \times V_1$, $B_r \times V_2$, respectively, then the restriction of $H_1 \circ H_2^{-1}$ to V^+ (resp. V^-) is an element of $\Lambda_{\omega_{1,\lambda}}(V^+)$ (resp. $\Lambda_{\omega_{1,\lambda}}(V^-)$). The normalizing application obtained in Theorem 2.10 is *not* however uniquely defined. Indeed, if H_1 is a normalizing application in $B_r \times (V_1 \setminus \{0\})$, then so is $H_1 \circ \phi$, for all $\phi \in \Lambda_{\omega_{1,\lambda}}(B_r \times V_1)$. In other words, the normalizing application is uniquely defined up to the composition with an element of $\Lambda_{\omega_{1,\lambda}}(V)$. This is why the change $H_1 \circ H_2^{-1}$ is *not* the identity in general. The change is indeed an *asymptotic expansion of the identity.* More precisely, it is an element of $\Lambda_{\omega_{1,\lambda}}(V_1)$. The special case in which the gluing of the leaves is the identity is exactly the case where the formal normal form is holomorphically conjugate to Dulac's normal form.

Proof of Proposition 2.1. As already mentioned, the sector change $\phi = H_1 \circ H_2^{-1}$ is an element of $\Lambda_{\omega_{1,\lambda}}$. Indeed Theorem 2.10 guarantees that conditions 1 and 2 are fulfilled. As to the third condition, we notice that H_i $(i = 1, 2)$ are such that:

$$dH_i(X) = X_{1,\lambda}(H_i)$$

where X is the given vector field (i.e. $X = y_1(1 + \lambda y_2) + y_2 R(y_1, y_2)\partial/\partial x + y_2^2 \partial/\partial y$) and $X_{1,\lambda}$ is its correspondent formal normal form (i.e. $X_{1,\lambda} = y_1(1 + \lambda y_2)\partial/\partial x + y_2^2\partial/\partial y$). Thus,

$$\begin{aligned} d(H_1 \circ H_2^{-1})(X_{1,\lambda})(H_1 \circ H_2^{-1})^{-1} &= dH_1 \circ dH_2^{-1}(X_{1,\lambda})H_2 \circ H_1^{-1} \\ &= dH_1 \circ dH_2^{-1} \circ dH_2(X) \circ H_1^{-1} \\ &= dH_1(X) \circ H_1^{-1} \\ &= X_{1,\lambda}(H_1) \circ H_1^{-1} \\ &= X_{1,\lambda}\,. \end{aligned}$$

The map $\phi(y_1, y_2) = (y_1 + b_0(y_2) + \sum_{i=1}^{\infty} b_i(y_2) y_1^i, y_2)$ belongs to $\Lambda_{\omega_{1,\lambda}}(V)$, if it is asymptotic to the identity and preserves the foliation. If we think in terms of vector fields the last condition implies that:

$$d\phi(X_{1,\lambda}) = X_{1,\lambda}(\phi)\,, \tag{2.31}$$

while the first one is satisfied if and only if the functions $b_j(y_2)$ are asymptotic to the zero function when y_2 tends to $0 \in \mathbb{C}$. The left hand side of Equation 2.31 is given by

$$y_1(1 + \lambda y_2)(1 + \sum_j j b_j(y_2) y_1^{j-1}) + y_2^2 b_0'(y_2) + y_2^2 \sum_j b_j'(y_2) y_1^j$$

while the right one is given by

$$\Big[y_1 + b_0(y_2) + \sum_j b_j(y_2) y_1^j\Big](1 + \lambda y_2)\,.$$

The equality between those expression leads us to the following ordinary differential equation:

$$b_j'(y_2) y_2^2 + b_j(y_2)(j-1)(1 + \lambda y_2) = 0\,.$$

whose solution is given by

$$b_j(y_2) = c_j y_2^{(1-j)\lambda} \exp\left(\frac{j-1}{y_2}\right),$$

where c_j is the initial data. Therefore unless $c_j = 0$ we have

$$\begin{cases} \lim\limits_{\Re(y_2)\to 0^+} b_j(y_2) = \infty \\ \lim\limits_{\Re(y_2)\to 0^-} b_j(y_2) = 0 \end{cases}$$

for all $j > 1$, where by $\Re(y_2)$ we mean the real part of y_2. We have therefore that, for $j > 1$, $b_j(y_2)$ is asymptotic to the null function if and only if $y_2 \in V^-$ or $c_j = 0$. This is equivalent to say that $c_j = 0$ on V^+ for all $j > 1$, i.e. that g^+ takes the form

$$g^+(y_1, y_2) = (y_1 + b_0(y_2), y_2)$$

In the same way we notice that

$$\begin{cases} \lim\limits_{\Re(y_2)\to 0^+} b_0(y_2) = 0 \\ \lim\limits_{\Re(y_2)\to 0^-} b_0(y_2) = \infty \end{cases}$$

Therefore or $y_2 \in V^+$ or $c_0 = 0$ in order to b_0 to be asymptotic to the null function. This implies that b_0 vanishes identically on V^-, i.e. that g^- is tangent to the identity.
■

Since each leaf of $X_{1,\lambda}$ is parametrized by a constant $c \in \mathbb{C}$ (Equation 2.30) the diffeomorphismes g^-, g^+ can be expressed in terms of the parametrization by those constants. In this way Proposition 2.1 says that

$$\begin{cases} g^+(c) = c + a_0 \\ g^-(c) = c + \sum_{i=2}^{\infty} a_i c^i \end{cases}$$

Now given two saddle-node foliations that are holomorphically conjugate, it is interesting to work out the relation between their sector changing diffeomorphisms (which in the $p = 1$ case, we denoted by g^+ and g^-). Indeed, we are going to see that if the saddle-nodes are holomorphically conjugate, then their sector changing diffeomorphisms are also conjugate by an automorphism of the leaf space. The converse is also true, though it will not be fully proved.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be holomorphically conjugate foliations given by 1-forms ω_1 and ω_2, respectively. In other words, there exists a diffeomorphism H that takes the leaves of $\mathcal{F}_1$ in leaves of $\mathcal{F}_2$. There are also formal changes of coordinates h_1 and h_2 that take ω_1 and ω_2, respectively in formal normal forms $\tilde{\omega}_1$ and $\tilde{\omega}_2$. As was seen in the earlier discussion, though h_1 (resp. h_2) is not analytic on a neighborhood of $0 \in \mathbb{C}$, there are sectors in which the series does converge. In the case $p = 1$ there are functions g_1^+, g_1^- (resp. g_2^+, g_2^-) that glue together the leaves of the sectors V^+ and V^-.

We shall define the following equivalence relation: two diffeomorphisms f and $\tilde{f}$ are said to be equivalent (and we write $f \sim \tilde{f}$) if there exists $\sigma \in \mathrm{Diff}(\mathbb{C}, 0)$ with $\sigma'(0) = 1$ such that:

$$\tilde{f} = \sigma^{-1} \circ f \circ \sigma\,.$$

In other words, conjugate functions belong to the same equivalence class. In this sense, if the foliations are holomorphically conjugate, the transition function g_1^+ (resp. g_1^-) is equivalent to g_2^+ (resp. g_2^-). Indeed, notice that the function $\sigma = h_2 \circ H \circ h_1^{-1} \in \mathit{Diff}(\mathbb{C}, 0)$ is an automorphism of the leaf space (which is isomorphic to a neighborhood of $0 \in \mathbb{C}$ as observed in Formula 2.30) tangent to the identity.

Denoting by σ_+ the restriction of σ to V^+ and by σ_- the restriction of σ to V^- it follows that $g_1^+ = \sigma_+^{-1} \circ g_2^+ \circ \sigma_+$ and $g_1^- = \sigma_-^{-1} \circ g_2^- \circ \sigma_-$.

Summarizing, the analytic type of saddle-nodes with $p = 1$ are in correspondence with the conjugacy class of the diffeomorphisms g^+ and g^-. Since the conjugacy class of translations is obvious, we can think that the information is totally encoded in the conjugacy of the diffeomorphism g^- tangent to the identity. It is then natural to study the moduli space of diffeomorphisms of $(\mathbb{C}, 0)$ tangent to the identity so as to have concrete invariants for saddle-node singularities.

Automorphisms of $(\mathbb{C}, 0)$ with identity linear part

Now we will concentrate on the study of the automorphisms σ of $(\mathbb{C}, 0)$ tangent to the identity that were used in the above definition of equivalence relation. The description of the moduli spaces is independently due to Ecalle and Voronin. In what follows we shall follow Voronin construction for the prototypical case $p = 1$.

Precisely, we shall give an analytic classification of the mappings of the set $\mathcal{A} = \{f \in \mathrm{Diff}(\mathbb{C}, 0)\ ;\ f(z) = z + az^2 + \cdots,\ a \neq 0\}$. As usual, we define two mappings f_1, $f_2 \in \mathcal{A}$ to be equivalent if and only if there exists a holomorphic diffeomorphism H such that $H \circ f_1 = f_2 \circ H$. Here we describe these classes of equivalence.

To begin with we give some Analysis results and definitions that will be used in this section.

Definition 2.5 *A homeomorphic mapping $f : \Omega \to \mathbb{C}$ of a domain $\Omega \subset \mathbb{C}$ is said to be quasiconformal if*

$$|f_{\bar{z}}| \le k|f_z|$$

for $k < 1$ almost everywhere.

The function $h_f = f_{\bar{z}}/f_z$ is called the *characteristic* of the quasiconformal map f, and the quantity

$$K_f(z_0) = \lim\,\sup_{r\to 0} \frac{\sup_{|z-z_0|=r} |f(z) - f(z_0)|}{\inf_{|z-z_0|=r} |f(z) - f(z_0)|}\,.$$

is called the *quasiconformal deviation* of the map f at the point z_0. Notice that if g is a quasiconformal map with characteristic $h_g = h_f$, then $g \circ f^{-1}$ is conformal.

Proposition 2.2 *A map is quasiconformal in Ω if and only if $K_f(z_0) < \infty$ for all $z_0 \in \Omega$ and $K = \|K_f\|_{L_\infty(\Omega)} < \infty$.*

Theorem 2.11 (Measurable Riemann Theorem) *For any measurable function h such that $\|h\|_{L_\infty} < 1$, there exists a quasiconformal map f of the plane $\mathbb{C}$ onto itself, having the function h as its characteristic $h = h_f$.*

Next, we consider the specific case of the function $f_0(z) = z/(1-z)$ in $\mathcal{A}$. Notice that the inversion $A_0(z) = -1/z$ conjugates f_0 conformally to the translation $T(z) = z+1$ on $\mathbb{C}^*$, that is,

$$A_0 \circ f_0 = T \circ A_0 \,. \tag{2.32}$$

Next we are going to see that the last remark holds in general. Namely for every function f belonging to $\mathcal{A}$, there always exist certain domains in $\mathbb{C}$ where f is *quasiconformally* conjugate to T. We prove this by using the next two results.

Theorem 2.12 (A. Shcherbakov) *Let $f \in \mathcal{A}$, $f_0(z) = z/(1-z)$. For every $\varepsilon > 0$ one can find δ and a homeomorphism $H(z) = z + h(z)$ of the disk K_δ (of radius δ) onto itself such that:*

$$\begin{aligned} H \circ f_0 &= f \circ H \text{ on } K_\delta && (2.33)\\ |h(z_1) - h(z_2)| &< \varepsilon |z_1 - z_2|,\ z_j \in K_\delta && (2.34) \end{aligned}$$

This theorem basically asserts that every function tangent to the identity is, in a sufficiently small disk, conjugate to f_0 by a Lipschitz mapping with Lipschitz constant close to 1. The proof of the theorem amounts to some finer estimates involving the Fatou coordinates [Car-G].

Lemma 2.5 *Suppose that for the homeomorphism $H(z) = z+h(z)$, Estimate (2.34) holds for $\varepsilon < 1$, then H is quasiconformal in K_δ.*

Proof. This lemma follows immediately from Proposition 2.2. Indeed, under these conditions, the quasiconformal deviation of H, $K(z_0)$ is such that $K(z_0) \leq (1+\varepsilon)/(1-\varepsilon)$ for $z \in K_\delta$. ■

Proposition 2.3 *For each $f \in \mathcal{A}$ there exist domains $R, L \subseteq \mathbb{C}$ and a quasiconformal mapping G defined on $R \cup L$, satisfying*

- $G \circ T = f \circ G$ *on* R,
- $G \circ T^{-1} = f^{-1} \circ G$ *on* L.

Proof. Fix ε such that $0 < \varepsilon < 1$. From Theorem 2.12 there exists a homeomorphism $H : K_\delta \to K_\delta$, conjugating f and f_0 as in (2.33). Moreover, this homeomorphism is quasiconformal by Lemma 2.5.

Let $R = \{z \in \mathbb{C};\ |\mathrm{Im}\, z| > c\,,\ \mathrm{Re} > c\}$ for a fixed complex number $c = 1/\rho$ with $\rho < \delta$. The mapping $G = H \circ A_0^{-1}$ defined on R is well-defined and quasiconformal, being a composition of a quasiconformal map with a conformal one. Set $\Omega_1 = G(R)$. It follows from (2.32) and (2.33) that G is precisely the conjugacy we were looking for. In conclusion, f is quasiconformally conjugate to a translation.

Moreover, since $T(R) \subset R$ it follows that Ω_1 is invariant by f, i.e. $f(\Omega_1) \subset \Omega_1$. Analogously, we can define a domain Ω_2 that is invariant by f^{-1}. More precisely, $\Omega_2 = G(L)$ where $L = \{z \in \mathbb{C};\ |\mathrm{Im}\, z| > c\,,\ \mathrm{Re} < -c\}$ with c as before. Naturally,

f^{-1} is defined on L and is quasiconformally conjugate to the translation T^{-1}. Thus establishing the proposition. ■

Equivalently, one may define the domains R and L so as to make their boundaries smooth (refer to [Car-G]). Under these circumstances $A_0^{-1}(R)$ is the well-known cardioid-shaped region (cf. Section 3.2.2). The reader may keep this figure in mind whenever we refer to R and L as it may help intuitively.

The next lemma gives analytic coordinates A_1 and A_2 defined on the domains Ω_1 and Ω_2. In what follows, we can trace a parallel between the saddle-node case, analyzed in the previous section, and the gluing of the domains Ω_1 and Ω_2. In this sense, we are still considering conjugacies which are not defined on a full neighborhood of the origin. In the saddle-node case, we have sectorial normalizations that are provided by Hukuara-Kimura-Matuda Theorem. The analogue of this theorem in the present case is the lemma below.

Lemma 2.6 (Basic Lemma) *Let R and Ω_1 be as before. Set $T(z) = z + 1$ and consider a quasiconformal homeomorphism G_1 of the domain R onto Ω_1. Let f be analytic in Ω_1 such that*

$$G_1 \circ T = f \circ G_1 \,. \tag{2.35}$$

Then there exists an analytic mapping $A_1 : \Omega_1 \to \mathbb{C}$ satisfying

1. *A_1 is univalent in Ω_1.*
2. *$A_1 \circ f = T \circ A_1$.*
3. *If A_1' is another analytic mapping on Ω_1 verifying conditions* 1 *and* 2*, then there exists $c \in \mathbb{C}$ such that $A_1' = A_1 + c$ on Ω_1.*

Notice that an analogous lemma may be formulated for the existence of the analytic mapping $A_2 : \Omega_2 \to \mathbb{C}$ with the obvious modifications.

Proof. As already seen, $f \in \mathcal{A}$ is conjugated to a translation. What this lemma asserts is that the conjugating homeomorphism is, in fact, analytic.

The quotient space R/T is conformally equivalent to the punctured plane $\mathbb{C}^*$, where $\pi_0 : R \to \mathbb{C}^*$ is the projection. Since G_1 is a homeomorphic mapping satisfying (2.35) the orbits of f are discrete and $\Omega_1/f = S$ is a Riemann Surface. Let $\tilde{\pi} : \Omega_1 \to S$ be the projection of the quotient space. From (2.35), we obtain a quasiconformal homeomorphism $\tilde{G} : \mathbb{C}^* \to S$ between the quotient spaces.

Claim 1 : If there exists a conformal mapping $B : S \to \mathbb{C}^*$ then there is an analytic mapping $A_1 : \Omega_1 \to \mathbb{C}$ verifying 1, 2 and 3.

Indeed, notice that Ω_1 is a covering for $\mathbb{C}^*$ with holomorphic projection $\pi = B \circ \tilde{\pi}$. However, the universal covering of $\mathbb{C}^*$ is $\mathbb{C}$ with projection $\hat{\pi}$. Since $\mathbb{C}$ is a simply connected covering of $\mathbb{C}^*$ then there exists an inclusion $A : \Omega_1 \hookrightarrow \mathbb{C}$ such that $\pi = \hat{\pi} \circ A$. This mapping satisfies 1, 2 and 3 (details may be found in [Vo]).

Claim 2 : If there exists a quasiconformal mapping $\tilde{G} : \mathbb{C}^* \to S$, then S is conformally equivalent to $\mathbb{C}^*$.

Since $\tilde{G}^{-1} : S \to U$ is a quasiconformal mapping from S to a domain U, the characteristic h of its inverse is independent on the choice of a local parameter on S, and $\|h\|_{L_\infty(U)} = k < 1$. Setting $h|_{\mathbb{C}\setminus U}$, h remains measurable and $\|h\|_{L_\infty(U)} = k$. By Theorem 2.11 there exists a quasiconformal homeomorphism F of $\mathbb{C}$ onto itself with characteristic h and we may suppose that $F(0) = 0$. So that the map $B = F \circ \tilde{G}^{-1} : S \to \mathbb{C}^*$ is conformal. ■

Now we analyze the functions that change the sector, i.e. the analogous to the functions $g_\pm$ in the saddle-node case.

Let Φ_+ (resp. Φ_-) be the restriction of $A_2 \circ A_1^{-1}|_{V_+}$ (resp. $A_2 \circ A_1^{-1}|_{V_-}$), where $V_+ = \Omega_1 \cap \Omega_2|_{\{z:\, \Im(z)>0\}}$ and $V_- = \Omega_1 \cap \Omega_2|_{\{z:\, \Im(z)<0\}}$ ($\Im(z)$ denotes the imaginary part of z).

Remark 2.1 It is not obvious, but with some effort an expression for $\Phi_\pm$ may be obtained. More precisely, $\Phi_\pm(z) = z + \sum_{k\geq 0} c_k^\pm \exp(\pm 2\pi i k z)$ (cf. [Vo]).

To each $f \in \mathcal{A}$ we may associate a pair $\Phi_\pm$ of holomorphic functions. Since we are interested in the classes of conjugacy of the elements of $\mathcal{A}$, it is natural to work out the relations between the maps $\Phi_\pm$ and $\tilde{\Phi}_\pm$ associated to conjugate diffeomorphisms f and $\tilde{f}$, respectively. The advantage of the preceding construction lies in the fact that the corresponding transition maps are related in a particularly simple way. Indeed, by Lemma 2.6 there exists analytic mappings A_1 and A_2 such that:

$$\begin{aligned} A_1 \circ f &= T \circ A_1 \\ A_2 \circ f^{-1} &= T^{-1} \circ A_2 \,. \end{aligned}$$

By assumption, $f = H_0^{-1} \circ \tilde{f} \circ H_0$ so that

$$\begin{aligned} A_1 \circ H_0^{-1} \circ \tilde{f} &= T \circ A_1 \circ H_0^{-1} \\ A_2 \circ H_0^{-1} \circ \tilde{f}^{-1} &= T^{-1} \circ A_2 \circ H_0^{-1} \end{aligned}$$

Suppose that $\tilde{A}_1$ is given by Lemma 2.6 for $\tilde{f}$. Then by item 3 of this same lemma, it follows that

$$A_1 \circ H_0^{-1} \equiv \tau_1 \circ \tilde{A}_1 \,.$$

Similarly,

$$A_2 \circ H_0^{-1} \equiv \tau_2 \circ \tilde{A}_2 \,,$$

where τ_1 and τ_2 are translations. Then

$$\begin{aligned} \Phi_\pm &= A_2 \circ A_1^{-1}|_{V_\pm} \\ &= \tau_2 \circ \tilde{A}_2 \circ H_0 \circ H_0^{-1} \circ \tilde{A}_1^{-1} \circ \tau_1^{-1} \\ &= \tau_2 \circ \tilde{\Phi}_\pm \circ \tau_1^{-1} \,. \end{aligned}$$

In conclusion, we have obtained the following

Theorem 2.13 *If f and $\tilde{f}$ belong to the same class of analytic equivalence then the transition functions $\Phi_{\pm}$ and $\tilde{\Phi}_{\pm}$ are conjugate by a translation.*

Remark 2.2 The reader may notice that, starting from the problem of classifying the saddle-nodes, we have iterated twice a procedure of "sectorial normalization". This might give the impression that no real progress was made in the second iteration towards a concrete description of a suitable moduli space. This is however not the case. In fact, in the saddle-node case, we have used sectorial normalizations provided by the Hukuara-Kimura Matuda Theorem to obtain diffeomorphisms g^{+}, g^{-} where conjugacy classes determine the analytic type of the saddle-node. The difficulty is that g^{-} is tangent to the identity but it is *not* uniquely determined. Only the conjugacy class of g^{-} in $\mathrm{Diff}(\mathbb{C}, 0)$ is canonical in the sense that it is uniquely determined. Thus, in a "concrete" comparison between two saddle-nodes we would be reduced to tell whether or not two "different" diffeomorphisms tangent to the identity are conjugate in $\mathrm{Diff}(\mathbb{C}, 0)$. The answer to this problem is by no means obvious. To tackle this new question, we again applied sectorial normalizations (this time provided by the Basic Lemma) to obtain new functions $\phi_{\pm}$. The advantage of this second normalization is that $\phi_{\pm}$ are "almost uniquely determined", in the sense that two of them are conjugate by a translation. In particular, it is easy to decide whether or not two of them define the same point in the corresponding moduli space.

2.3.2 Reduction of Singularities in Dimension 2

So far we have only been analyzing vector fields with simple singularities. However, as it will become clear throughout this section, there is a particularly effective way of dealing with higher order singularities. The Seidenberg Theorem basically asserts that by composing a finite number of blow-up applications, it is possible to reduce the order of an isolated singularity until we only obtain simple ones. This section is devoted to explain how it can be done.

The reader will notice that the present exposition is very strongly inspired in the treatment given in [M-M], which, in turn, is a blend between the original work of A. Seidenberg [Sd] and that of Ven den Essen.

Let us now fix notations and give a few definitions. Suppose that $\mathcal{F}$ is a singular holomorphic foliation associated to the holomorphic vector field X having an isolated singularity at $(0, 0)$. We can also think in terms of 1-forms. After all, the 1-form $\omega = Adx + Bdy$ and the vector field $X(x, y) = B(x, y)\frac{\partial}{\partial x} - A(x, y)\frac{\partial}{\partial y}$ define the same foliation $\mathcal{F}$. The *eigenvalues* of $\omega = Adx + Bdy$ at $(0, 0)$ are defined to be the eigenvalues λ_1, λ_2 of X at the same point.

Let A_n (resp. B_n) denote the homogeneous component of degree n of the Taylor series of A (resp. B) centered at the origin. Let k be the degree of the first non-trivial homogeneous component of the 1-form $\omega = Adx + Bdy$. The blow-up of ω in the coordinates (x, t) is given by:

$$\pi^*(\omega) = [A_k(1, t) + tB_k(1, t) + x(\tilde{A}(x, t) + t\tilde{B}(x, t))]dx + x[B_k(1, t) + x\tilde{B}(x, t)]dt\,.$$

while in the coordinates (t, y) it is given by:

$$\pi^*(\omega) = y[A_k(t,1) + y\tilde{A}(t,y)]dt + [B_k(t,1) + tA_k(t,1) + y(\tilde{B}(t,y) + t\tilde{a}(t,y))]dy$$

where

$$\tilde{A}(x,t) = A_{k+1}(1,t) + xA_{k+2}(1,t) + x^2A_{k+3}(1,t) + \cdots = \frac{1}{x^k}\sum_{n>k} A_n(x,tx)\,.$$

We adopt an analogous notation for $\tilde{B}$.

Let $J^1_{(0,0)}(\omega) = A_1(x,y)dx + B_1(x,y)dy$, and denote by C_ω the subset of the exceptional divisor $E \simeq \mathbb{CP}(1)$ formed by the singularities of the "new" foliation $\widetilde{\mathcal{F}}$ (i.e. the foliation associated to $\tilde{\omega} = \pi^*\omega$). Notice that in the non-dicritical case ($A_1(1,t) + tB_1(1,t)$ is *not* identically zero) the set C_ω is given by the solutions of $A_1(1,t) + tB_1(1,t) = 0$. Denote by μ_c the order of the singularity $c \in C_\omega$.

Suppose that F is a mapping defined on $U \subset \mathbb{C}^2$ with a singularity at $(0,0) \in \mathbb{C}^2$. The *order* of F at 0, denoted by $\nu_0(F)$, is the degree of the first non-trivial homogeneous component of the Taylor series of F based at the origin. We may also define the order of a vector field (as was done in Section 2.4). If $X(x,y) = (F(x,y), G(x,y))$ is singular at $(0,0)$ the *order* of X is the minimum between the orders of F and G. Analogously one may define the *order* of a 1-form ω at 0.

Let f, g be two polynomials defined on $\mathbb{C}^2$. Consider two algebraic curves V and W associated, respectively, to f and g, i.e. consider

$$V = \{(x,y) \in \mathbb{C}^2;\ f(x,y) = 0\}, \qquad W = \{(x,y) \in \mathbb{C}^2;\ g(x,y) = 0\}\,,$$

and suppose that they intersect each other at $(0,0) \in \mathbb{C}^2$. Recall that if g is irreducible then there exists a local normalization (Puiseux Parametrization)

$$\gamma : t \mapsto (t^k, \sum_{n=m}^{\infty} a_n t_n)\,,$$

for a certain k and $m < k$. We are now able to introduce the notion of *intersection number* of two algebraic curves.

Definition 2.6 *If g is irreducible then the intersection number of V and W at $(0,0)$ is defined to be*

$$I(f,g;0) = \nu_0(f \circ \gamma)\,.$$

If g is not irreducible but $g = g_1^{\alpha_1} \ldots g_p^{\alpha_p}$, where $g_1, \ldots, g^p$ are irreducible, then the intersection number of V and W at $(0,0)$ is defined to be

$$I(f,g;0) = \sum_{i=1}^{p} \alpha_i I(f,g_i;0)\,.$$

Finally, the intersection number of the 1-form $\omega = Adx + Bdy$ is defined to be

$$I_0(\omega) = I(A,B;0)\,.$$

At this point we are able to state Seidenberg's Theorem.

Theorem 2.14 (Seidenberg) *Let $\mathcal{F}$ be a singular holomorphic foliation associated to the* 1*-form $\omega = Adx + Bdy$ defined on an open set $U \subseteq \mathbb{C}^2$ admitting an isolated singularity at $(0,0)$. There exists a proper analytic application $\pi : V \to U$ (obtained as a composition of blow-ups) of a complex* 2*-dimensional manifold V onto U, such that:*

- $\pi^{-1}(0,0) = E$*, where E is the exceptional divisor of V;*
- $\pi : V \setminus E \to U \setminus \{(0,0)\}$ *is a holomorphic diffeomorphism;*
- $\nu_p(\pi^*(\omega)) \leq 1$ *for all $p \in V$*

In other words, the proper transform of $\mathcal{F}$ is defined on a complex manifold V and is such that all of its points $p \in V$ are either regular or simple singularities.

Proof. Suppose that $(0,0)$ is the only singularity of ω in U. Let k be the order of ω at $(0,0)$. If $k > 1$, we use the blow-up procedure, described in Section 2.4. In other words, we consider the pull-back of ω by the proper mapping $\pi : \widetilde{\mathbb{C}}^2 \to \mathbb{C}^2$ given by $\pi(x,t) = (x, tx)$. Let $\pi^{-1}(0,0) = E$ denote the exceptional divisor. In coordinates (x,t) the blow-up of ω is given by

$$\tilde{\omega} = [A_k(1,t) + tB_k(1,t) + x(\tilde{A}(x,t) + t\tilde{B}(x,t))]dx + x[B_k(1,t) + x\tilde{B}(x,t)]dt \tag{2.36}$$

with $\tilde{A}$, $\tilde{B}$ as above.

As previously seen, the behavior of $\tilde{\omega}$ varies significantly on the neighborhood of E depending on whether or not $A_k(1,t) + tB_k(1,t)$ is identically zero. By analyzing these two different cases, it is not hard to obtain equalities relating the intersection number of ω and its order at $(0,0)$ (refer to [M-M] for details). More specifically,

- If $A_k(1,t) + tB_k(1,t)$ is *not* identically zero, i.e. if the exceptional divisor is invariant under the foliation, then

$$I_0(\omega) = k^2 - k + 1 + \sum_{c \in E} I_c(\tilde{\omega}) \,. \tag{2.37}$$

- If $A_k(1,t) + tB_k(1,t) \equiv 0$, then

$$I_0(\omega) = k^2 + k - 1 + \sum_{c \in E} I_c(\tilde{\omega}) \,. \tag{2.38}$$

  This corresponds to the dicritical case, i.e. to the case where the leaves of the foliation $\tilde{\mathcal{F}}$ associated to $\tilde{\omega}$ are regular and generically transverse to E.

There are two possibilities, either the 1-form $\tilde{\omega}$ has only simple singularities and regular points on $\pi^{-1}(U)$ and the theorem is proved, or there still are points $c \in \pi^{-1}(U)$, such that $\nu_c(\tilde{\omega}) > 1$. So, let us assume that the second case occurs. We first set

$$V_1 = \pi^{-1}(U)\,, \qquad \pi_1 = \pi|_{V_1} : V_1 \to U.$$

Next, we simultaneously blow-up all the points $c \in E \subset V_1$ such that $\nu_c(\tilde{\omega}) > 1$. Let $\pi_2 : V_2 \to V_1$ be the resulting application corresponding to these blow-ups. We then define

$$\pi^2 = \pi_1 \circ \pi_2 : V_2 \to U\,.$$

Inductively we construct applications $\pi_i : V_i \to V_{i-1}$, by blowing-up all the points c of V_{i-1} such that $\nu_c((\pi^{i-1})^*(\omega)) > 1$, where

$$\pi^{i-1} = \pi_1 \circ \cdots \circ \pi_{i-1}\,.$$

The Equations (2.37) and (2.38) guarantee that this procedure is finite. Indeed, despite the two different behaviors the blown-up foliation $\widetilde{\mathcal{F}}$ may assume, in both cases the intersection number I_c decreases if $k > 1$. So if we repeat this procedure sufficiently many times, there will be a large enough i such that for every c on V_i, $\nu_c((\pi^i)^*(\omega)) \leq 1$. ∎

Now we shall treat the case of a foliation $\mathcal{F}$ with only simple singularities. By performing additional blow-ups it is possible to obtain a simpler expression for the 1-form $\omega = Adx + Bdy$ associated to $\mathcal{F}$. This corresponding form cannot be further simplified by extra blow-ups, and in this sense they are “final” or “irreducible”.

Theorem 2.15 *Consider the singular holomorphic foliation associated to the* 1*-form* $\omega = Adx + Bdy$ *defined on* $U \in \mathbb{C}^2$ *with an isolated singularity at* $(0,0)$*. Let* $\pi : V \to U$ *the blow-up application obtained by Theorem 2.14. Then for every* $p \in V$ *where* $\nu_p(\pi^*(\omega)) = 1$*, there exists a coordinate chart* (u,v) *centered on* p *such that*

$$\pi^*(\omega) = (\lambda_1 + \text{h.o.t})vdu - (\lambda_2 + \text{h.o.t.})udv\,, \tag{2.39}$$

where $\lambda_1.\lambda_2 \neq 0$*, and* λ_1/λ_2*,* λ_2/λ_1 *do not belong to* $\mathbb{N}$*, or*

$$\pi^*(\omega) = (v + \text{h.o.t.})du\,. \tag{2.40}$$

Let us begin our approach to Theorem 2.15 by stating the following result:

Lemma 2.7 *Suppose that the* 1*-form* $\omega = Adx + Bdy$ *is non-dicritical. If there exists* $c \in \mathbb{CP}(1)$ *such that* $\mu_c = 1$*, then* $J^1(\tilde{\omega}_c) = \tilde{A}_1dx + \tilde{B}_1dy \neq 0$*. In other words, the blow-up of* ω *at* c*,* $\tilde{\omega}_c$*, does not raise the order* $\mu_{\pi^{-1}(c)}$*. Moreover,* $\tilde{\omega}_c$ *has a non-zero eigenvalue.*

Proof. We may suppose that the order of ω at $(0,0)$ is $k = 1$. According to the eigenvalues of ω, there are 5 different situations that should be analyzed separately:

1. $\lambda_1 = \lambda_2 = 0$;
2. $\lambda_1 = \lambda_2 = \lambda \neq 0$;
3. $\lambda_1 \neq \lambda_2$, $\lambda_1.\lambda_2 \neq 0$ and λ_1/λ_2, λ_2/λ_1 belong to $\mathbb{N}$;
4. $\lambda_1 \neq \lambda_2$, $\lambda_1.\lambda_2 \neq 0$ and λ_1/λ_2, λ_2/λ_1 do *not* belong to $\mathbb{N}$;
5. $\lambda_1 \neq \lambda_2$, $\lambda_1.\lambda_2 = 0$.

Notice that the cases 4 and 5 correspond precisely to the situations reflected by Equations (2.39) and (2.40), respectively. Now we show that by blowing up the manifold on certain points, the 3 previous cases are reduced to cases 4 or 5.

Case 1

In this situation, the jacobian matrix of the vector field $X = (B, -A)$ at $(0,0)$ is similar to

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Therefore, $J^1_{(0,0)}(\omega) = ydy$, so that the set C_ω is formed by the point $c = (1,0)$ and $\mu_c = 2$.

Now we blow-up the foliation at c, obtaining

$$\tilde{\omega}_c = [t^2 + x(t\tilde{b}(x,t) + \tilde{a}(x,t))]dx + x[t + x\tilde{b}(x,t)]dt\,,$$

Suppose that $A_2(1,t) = \alpha_1 + \alpha_2 t + \alpha_3 t^2$.

Notice that the order of $\tilde{\omega}$ may be 2 or 1 depending on whether or not α_1 is equal to zero. We analyze the two possibilities separately:

- **(1.a)** $\alpha_1 \neq 0$

  In this case, $\nu(\tilde{\omega}) = 1$ and to simplify the notation we denote $\tilde{\omega}$ by:

  $$\eta = [y^2 + x(y\tilde{b}(x,y) + \tilde{a}(x,y))]dx + x[y + x\tilde{b}(x,y)]dy\,,$$

  Notice that C_ω is given by the equation

  $$\alpha_1 x^2 = 0,$$

  hence formed by the point $c = (0,1)$ with $\mu_c = 2$. Now we blow-up at c and in the coordinate chart (t,y), $t = x/y$ to obtain

  $$\tilde{\eta}_c = y[\alpha_1 t + y(1 + \cdots)]dt + [\alpha_1 t^2 + y(2t + \cdots)]dy.$$

  Therefore, $C_{\tilde{\eta}_c}$ is given by the equation

  $$ty(3y + 2\alpha_1 t) = 0,$$

  and hence it contains only simple points. By applying Lemma 2.7, we conclude that in these coordinates $\lambda_1 \neq 0$. Therefore we have reduced ω to a 1-form which does not belong to **Case** 1.

- **(1.b)** $\alpha_1 = 0$

  Here, $\nu(\tilde{\omega}) = 2$, and as before, denote $\tilde{\omega}$ by η. The set C_η is given by
  $$x(2y^2 + \gamma xy + \delta x^2) = x(y - a_1 x)(y - a_2 x) = 0\,,$$
  for certain constants γ, δ, a_1 and a_2.

  – If $a_1 \neq a_2$ we apply Lemma 2.7 and the conclusion is the same as before.

  – If $a_1 = a_2 = a \neq 0$ then
  $$\tilde{\eta} = [(t-a)^2 + x(\cdots)]dx + x[t + x(\cdots)]dt\,.$$
  Hence the jacobian matrix associated to $\tilde{\omega}$ is $\begin{pmatrix} * & * \\ 0 & 2a \end{pmatrix}$, therefore admitting a non-zero eigenvalue.

  – If $a = 0$ then the point $c = (1,0) \in C_\eta$, has multiplicity $\mu_c = 2$ and
  $$\tilde{\eta}_c = [2t^2 + x(\cdots)]dx + x[t + x(\cdots)]dt$$
  So, $\tilde{\eta}$ is still of the same type as η. Notice that by Theorem 2.14 $\nu(\tilde{\eta}) = 1$. Therefore this possibility leads to **Case (1.a)**.

In conclusion, there always exists a way to blow-up ω so that both of its eigenvalues are different from *zero*.

Case 2

Without loss of generality we may suppose that $\lambda = 1$. Hence there are basically two possibilities to be considered:

- The jacobian matrix of the vector field X at $(0,0)$ is similar to
  $$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
  In this situation, $J^1_{(0,0)}(\omega) = ydx - xdy$. It is therefore the dicritical case. By blowing-up ω at each $c \in \mathbb{CP}(1)$, notice that $\tilde{\omega}_c$ has no singularities ($\nu_p(\tilde{\omega}_c) = 0$ for all $p \in V$).

- The jacobian matrix of the vector field X at $(0,0)$ is similar to
  $$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
  Here, $J^1_{(0,0)}(\omega) = -xdx + (x+y)dy$, and the set C_ω is the point $c = (1,0)$ with $\mu_c = 2$. By blowing-up ω at c we obtain:
  $$\tilde{w}_c = [t^2 + x(\cdots)]dx + x[1 + t + x(\cdots)]dt$$
  Notice that the jacobian matrix associated to $\tilde{\omega}$ at c is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. This corresponds to **Case** 5, so that ω is reduced to Equation (2.40).

Case 3

In this case, $J^1_{(0,0)}(\omega) = -\lambda_2 y dx + \lambda_1 x dy$. We may suppose that $\lambda_1 = 1$ and $\lambda_2 = n$. So, ω may be written as

$$\omega = [-ny + (\cdots)]dx + [x + (\cdots)]dy\,.$$

Notice that there are exactly two points on C_ω, namely $c_1 = (1,0)$ and $c_2 = (0,1)$. Blowing-up ω at c_1 on the coordinate chart (x,t), $t = y/x$ we obtain

$$\tilde{\omega}_{c_1} = [t(1-n) + x(\cdots)]dx + x[1 + x(\cdots)]dt$$

so that the jacobian matrix is:

$$\begin{pmatrix} \lambda_1 & 0 \\ * & \lambda_2 - \lambda_1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & n-1 \end{pmatrix}$$

Therefore, we have reduced ω to a 1-form having eigenvalues $\tilde{\lambda}_1 = 1$ and there are three possibilities for $\tilde{\lambda}_2$:

- If $n = 0$ then $\tilde{\lambda}_2 = -1$ and the reduction of ω belongs to **Case** 4.
- If $n = 1$ then $\tilde{\lambda}_2 = 0$ and the reduction of ω belongs to **Case** 5.
- If $n > 1$ then $\tilde{\lambda}_2 = n - 1$ and by blowing-up $n-1$ times, the reduction of ω belongs to **Case** 2.

Now let us analyze the blowing-up of ω at c_2 on the coordinate chart (t,y), $t = y/x$ where

$$\tilde{\omega}_{c_2} = y[-n + y(\cdots)]dt + [t(1-n) + y(\cdots)]dy\,.$$

Hence the jacobian matrix is given by:

$$\begin{pmatrix} \lambda_1 - \lambda_2 & * \\ 0 & \lambda_2 \end{pmatrix} \sim \begin{pmatrix} 1-n & 0 \\ 0 & n \end{pmatrix}$$

- If $n = 0$ then $\tilde{\lambda}_1 = 1$ and $\tilde{\lambda}_2 = 0$, hence the reduction of ω belongs to **Case** 5.
- If $n = 1$ then $\tilde{\lambda}_1 = 0$ and $\tilde{\lambda}_2 = 1$, hence the reduction of ω belongs to **Case** 5.
- If $n > 1$ then $\tilde{\lambda}_1 < 0$ and $\tilde{\lambda}_2 > 0$, hence the reduction of ω belongs to **Case** 4.

This completes the proof of the theorem. ∎

2.3.3 Existence of First Integrals

To illustrate the applications of the preceding techniques on concrete problems about singularities of foliations, we shall provide in this section a detailed proof of a fundamental result due to J.-F Mattei and R. Moussu [M-M] concerning the existence of first integral for those singularities.

Recall that a holomorphic *first integral* of a holomorphic foliation $\mathcal{F}$ is a non-constant holomorphic function $f : (\mathbb{C}^2, 0) \to \mathbb{C}$ that is constant when restricted to the leaves of $\mathcal{F}$. If ω is a 1-form defining $\mathcal{F}$ then f is such that $\omega \wedge df = 0$. Equivalently, the leaves of $\mathcal{F}$ are the irreducible components of the level surfaces of f.

This section is devoted to the topological characterization of foliations admitting a holomorphic first integral, which is itself the main result of [M-M].

Let $\mathcal{F}$ be a holomorphic foliation defined on an open set $U \subset \mathbb{C}^2$, with an isolated singularity at $(0, 0)$. Suppose that f is a holomorphic first integral for $\mathcal{F}$. Then the following conditions hold:

1. Only a *finite* number of leaves of $\mathcal{F}$ accumulates on $(0, 0)$.

2. The leaves of $\mathcal{F}$ are closed as subsets of $U \setminus \{(0, 0)\}$.

Indeed, if this function exists then the set $f^{-1}(0)$ is the set of the separatrizes of $\mathcal{F}$. Recall that a separatrix is simply an irreducible analytic curve invariant by $\mathcal{F}$, which contains the singularity. Now f can have only finitely many irreducible factors, i.e. $f = g_1^{k_1} \dots g_n^{k_n} H$ for some positive integer constants $k_1, \dots, k_n$ and holomorphic functions $g_1, \dots, g_n, H$ such that $H(0, 0) \neq 0$. Hence the number of separatrizes must also be finite, as these are given by the equations $\{g_i = 0\}$. Clearly this is the contents of Condition 1.

Since f is constant when restricted to the leaves, Condition 2 also holds. In particular, a leaf L of $\mathcal{F}$ that is not contained in the set $f^{-1}(0)$ cannot accumulate on a separatrix. This last remark will often be used in the sequel.

Conversely, a foliation satisfying Conditions 1 and 2 admits a first integral. This is the contents of the next theorem that plays the main role in this section.

Theorem 2.16 (Mattei-Moussu [M-M]) *Consider the holomorphic foliation $\mathcal{F}$ defined on an open set $U \subset \mathbb{C}^2$ with an isolated singularity at $(0, 0)$. Suppose that it satisfies the following conditions:*

1. *Only a finite number of leaves of $\mathcal{F}$ accumulates on $(0, 0)$;*

2. *The leaves of $\mathcal{F}$ are closed on $U \setminus \{(0, 0)\}$.*

Then $\mathcal{F}$ has a holomorphic non-constant first integral $f : U \to \mathbb{C}$.

Before proving the theorem, consider two holomorphic foliations with isolated singularities, $\mathcal{F}$ and $\mathcal{F}'$ defined on a neighborhood U of the origin. We say that $\mathcal{F}$ and $\mathcal{F}'$ are *topologically equivalent* if there exists a homeomorphism $h : U \to U$, such

that $h(0,0) = (0,0)$ taking the leaves of $\mathcal{F}$ into the leaves of $\mathcal{F}'$. It follows from the above theorem that the existence of a holomorphic first integral obeys a topological criterion: if $\mathcal{F}$ admits one such first integral so does $\mathcal{F}'$.

Roughly speaking, the proof of Theorem 2.16 consists of three main steps. These are as follows:

Part 1: If $\mathcal{F}$ satisfies Conditions 1 and 2, then, by performing successive blow-ups, the only irreducible singularities found in the Seidenberg tree are those having two non-zero eigenvalues λ_1 and λ_2 verifying in addition $\lambda_1/\lambda_2 \in \mathbb{R}_-$. Thus the singularities belong to the Siegel domain.

Part 2: Study of $\mathcal{F}$ on a neighborhood of a singularity in the Siegel domain. In particular the characterization of those possessing a holomorphic first integral.

Part 3: Extension of the local "first integrals" obtained in **Part** 2 to a neighborhood of the exceptional divisor E. If $\tilde{f}$ denotes this extension, then $f = \pi_* \tilde{f}$ is the first integral we were looking for.

We will give a detailed approach to each part described above in the next 3 sections. Together, they will conclude the proof of the Mattei-Moussu Theorem.

Part 1: Analysis of Singularities in the Seidenberg Tree of $\mathcal{F}$

Let us now begin by analyzing the irreducible singularities obtained by the Seidenberg procedure. From now on we shall assume that $\mathcal{F}$ satisfies the assumptions of Theorem 2.16.

Proposition 2.4 *Let $\mathcal{F}$ be a foliation satisfying Conditions* 1 *and* 2*. Then the only irreducible singularities that exist in the Seidenberg tree of $\mathcal{F}$ are those having non-zero eigenvalues λ_1 and λ_2 such that the quotient λ_1/λ_2 belongs to $\mathbb{R}_-$.*

As seen in the previous section, every singularity of a given foliation $\mathcal{F}$ may be reduced by successive blow-ups so that the blown-up foliation $\widetilde{\mathcal{F}}$ only contains simple singularities with eigenvalues λ_1 and λ_2 which fall in one of the following cases:

- $\lambda_1.\lambda_2 = 0$ where at least one between λ_1, λ_2 is distinct from zero (the case of a saddle-node singularity).
- $\lambda_1.\lambda_2 \neq 0$ with neither λ_1/λ_2 nor λ_2/λ_1 being a positive integer.

Let us discuss the different possibilities separately.

Lemma 2.8 *The Seidenberg tree contains no saddle-nodes.*

Proof. Suppose for a contradiction that the statement is false. As we have seen in previous sections there are local coordinates (x, y) about a saddle-node singularity where the blown-up foliation $\widetilde{\mathcal{F}}$ is given by

$$\tilde{\omega}(x, y) = [x(1 + \lambda y^p) + yR(x, y)]dy - y^{p+1}dx \,.$$

Note that $\{y=0\}$ is invariant by the blown-up foliation $\widetilde{\mathcal{F}}$. Next let us consider the holonomy application of $\widetilde{\mathcal{F}}$ associated to a small loop contained in $\{y=0\}$ encircling the origin. We have already seen on Section 2.2.2 that the holonomy is given by $h(z) = z + z^{p+1} + \cdots$ for z on some local transverse section Σ.

From the dynamical structure of h (in particular the existence of Fatou coordinates) we see that there are infinitely many points whose orbit under the iteration of h accumulats on $0 \in \mathbb{C} \simeq \Sigma$. These points naturally correspond to distinct leaves of $\mathcal{F}$ accumulating on the regular part of $\{y=0\}$. Now we have two possibilities.

If $\{y=0\}$ is contained on the exceptional divisor, then it is projected onto $(0,0)$. Hence there are infinitely many leaves accumulating on the origin. This is not possible, for we are assuming Condition 1.

On the other hand, if $\{y=0\}$ is transverse to the exceptional divisor, then it is projected by π as a separatrix (recall that π is proper). There are infinitely many leaves accumulating on this separatrix, which again is impossible, since it contradicts Condition 2.

In conclusion the reduction of ω does not contain any saddle-node singularity. ■

Next we shall discuss irreducible singularities having two eigenvalues λ_1, λ_2 different from zero. Recall that such a singularity is called *hyperbolic* if $\lambda_1/\lambda_2 \in \mathbb{C} \setminus \mathbb{R}$.

Lemma 2.9 *There is no hyperbolic singularity in the Seidenberg tree.*

Proof. By the Poincaré Linearization Theorem (cf. Section 2.2.1) we may suppose that the vector field associated to $\widetilde{\mathcal{F}}$ is given in local coordinates by

$$\widetilde{X} = \lambda_1 x \frac{\partial}{\partial x} + \lambda_2 y \frac{\partial}{\partial y}$$

Again, $\{y=0\}$ is invariant by the foliation and, in order to study the behavior of the leaves close to $\{y=0\}$, we are going to consider its local holonomy.

Let $x(t) = \varepsilon e^{2\pi i t}$ be a small loop around the origin of $\mathbb{C}_x$. Let $\Sigma = \{(\varepsilon, y) : y \in \mathbb{C}\}$ be a transversal section to the leaf $\{y=0\}$ through the point $(\varepsilon, 0)$. The lift of the loop to the leaf through the point $(\varepsilon, y) \in \Sigma$ is described by the differential equation

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = 2\pi i \frac{\lambda_2}{\lambda_1} y$$

whose solution is given by $y(t) = y e^{2\pi i \lambda_2/\lambda_1 t}$. Hence the holonomy map turns out to be $h(y) = e^{2\pi i \lambda_2/\lambda_1} y$. However we have that $\Re(2\pi i \lambda_2/\lambda_1) \neq 0$ since λ_2/λ_1 is not in $\mathbb{R}$. Now by iterating h, we obtain

$$h^n(y) = r^n \alpha^n y\,.$$

where $r = e^{\Re(2\pi i \lambda_2/\lambda_1)}$ and $\alpha = e^{\Im(2\pi i \lambda_2/\lambda_1)}$. The absolute value of $h^n(y)$ is determined by the value of r. More specifically, if $r = e^{-2\pi \mathrm{Im}(\lambda_2/\lambda_1)} < 1$ then $|h^n(y)|$ tends to zero as n increases and, consequently the correspondent leaf accumulate on $\{y=0\}$. If $r > 1$ then $|h^n(y)|$ goes to zero as n goes to minus infinity. This also ensures that

our leaf accumulate on $\{y = 0\}$. However, Conditions 1 and 2 prevent this type of behavior from happening and we conclude that the reduced foliation does not contain this kind of singularities. ■

Lemma 2.10 *The Seidenberg tree does not contain singularities with non-zero eigenvalues such that $\lambda_1/\lambda_2 \in \mathbb{R}_+$.*

Proof. By Theorems 2.3 and 2.15, the vector field is still linearizable and $\{y = 0\}$ is invariant by the correspondent foliation. Notice that in this case the holonomy application will not help us to conclude that this singularity does not occur. In fact the holonomy map has the form $y \mapsto e^{2\pi i\theta}y$ for some $\theta \in \mathbb{R}$, i.e. the holonomy is conjugate to a rotation of $\mathrm{Diff}(\mathbb{C}, 0)$.

However, by following the radial lines connecting a point x_0 on the plane $\{y = 0\}$ to the origin, we see that the neighboring leaves accumulate on the origin. Indeed, fix a point $(x_0, y_0) \in \mathbb{C}^2$ and consider the lift of the radial line $x(t) = x_0e^{-t}$, $t \in \mathbb{R}_+$, contained in $\{y = 0\}$ to the leaf through (x_0, y_0). This lift is described by the differential equation

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = -\frac{\lambda_2}{\lambda_1}y$$

whose solution is given by $y(t) = y_0e^{-\lambda_2/\lambda_1 t}$. Since the quotient λ_2/λ_1 is positive it follows that $(x(t), y(t))$ goes to $(0, 0)$ as t goes to infinity. Ultimately, this implies that there are infinitely many leaves accumulating on $(0, 0)$. Again, such singularities do not appear on the blown-up foliation. ■

Finally, we have reached the conclusion that we may assume that the blown-up foliation $\widetilde{\mathcal{F}}$ only has singularities with eigenvalues such that $\lambda_1/\lambda_2 \in \mathbb{R}_-$. Before proceeding we only remark that:

Lemma 2.11 *All the irreducible components of the Seidenberg tree are invariant by the corresponding foliation.*

Proof. If it were not so there would be infinitely many separatrizes, contradicting our assumptions. ■

Remark 2.3 The argument used in the preceding lemma can be extended to show that a singularity has infinitely many separatrizes if and only if by performing a finite sequence of blow-ups, we arrive to an irreducible component of the total exceptional divisor which is *not* invariant by the corresponding foliation. This type of singularity is called *dicritical.*

The above lemmata leading to Proposition 2.4, corresponds to the first part of our proof.

Part 2: Existence of Local First Integrals

Let us take a closer look at the only singularities that appear in the Seidenberg tree, i.e. those having eigenvalues λ_1, λ_2 such that λ_1/λ_2 is in $\mathbb{R}_-$. We obtain from Lemma 2.2 that in local coordinates (x, y), the vector field is given by:

$$\lambda_1 x(1 + \text{h.o.t.})\partial/\partial x + \lambda_2 y(1 + \text{h.o.t.})\partial/\partial y\,, \tag{2.41}$$

for $\lambda_1/\lambda_2 \in \mathbb{R}_-$. First, what needs to be studied is whether or not this vector field is linearizable. More generally we would like to tell when two vector fields as above (with λ_1, λ_2 fixed) are conjugate. The next proposition due to J.-F. Mattei and R. Moussu is of fundamental importance for it allows us to restrict ourselves to understanding whether or not the corresponding *holonomy* is linearizable (conjugate).

Theorem 2.17 (Mattei-Moussu [M-M], [M]) *Assume that $\mathcal{F}_1, \mathcal{F}_2$ are given in coordinates (x, y) by Equation (2.41) with $\lambda_1/\lambda_2 \in \mathbb{R}_-$. Denote by h_1 (resp. h_2) the holonomy of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) relative to the axis $\{y = 0\}$. Then there is an analytic diffeomorphism ϕ defined on a neighborhood of $(0, 0) \in \mathbb{C}^2$ and conjugating $\mathcal{F}_1, \mathcal{F}_2$ if and only if there is an analytic diffeomorphism φ defined on a neighborhood of $0 \in \mathbb{C}$ and conjugating h_1, h_2.*

There are two markedly different cases to be considered according to whether or not λ_1/λ_2 is rational. Firstly, suppose that $\lambda_1/\lambda_2 = -m/n$ where $m, n \in \mathbb{N}$. In this case, the holonomy of $\mathcal{F}$ associated to a loop around the origin in the separatrix $\{y = 0\}$, is given by

$$h(z) = e^{-2\pi i n/m} z + \text{h.o.t.}\,,$$

where z is the parameter on a local section transverse to $\{y = 0\}$. This application is linearizable if and only if $h^m(z) = z$. It is obvious that if h is linearizable then $h^m(z) = z$. The converse is not obvious but can easily be verified by noticing that $f = (\frac{1}{m}\sum_{i=0}^{m-1} \lambda^{-i} h^i)^{-1}$ linearizes h. Note that when it is not linearizable (i.e. when $h^m(z) = z +$h.o.t.) we end up in the case already studied. Namely, we end up on the dynamics of the cardioid-shaped region analyzed on Section 2.2.2. Similarly to the saddle-node case, this cannot occur since otherwise we would have infinitely many leaves accumulating on $\{y = 0\}$, what contradicts our assumptions.

Hence, we may suppose that h is indeed linearizable. From Theorem 2.17, the vector field associated to $\widetilde{\mathcal{F}}$ admits the normal form, given in local coordinates (x, y) by

$$\widetilde{X} = mx\frac{\partial}{\partial x} - ny\frac{\partial}{\partial y}\,. \tag{2.42}$$

Let us now focus on the case where $\alpha = \lambda_1/\lambda_2$ is irrational. As before, the holonomy application is given by $h(z) = e^{2\pi i\alpha} z + \cdots$. However, since α is an irrational number, the difficulty of knowing whether or not h is linearizable increases considerably. The problem of finding out which are the irrational values of α that make the holonomy h linearizable is known as the "Siegel Problem". In fact, for certain values of α, it is possible to obtain local diffeomorphisms h as above that are non-linearizable.

Remark 2.4 Note that it is not difficult to obtain a *formal* conjugacy f that linearizes any such h. More precisely, by formally solving $f^{-1} \circ h \circ f(z) = \lambda z$, one obtains $f(z) = \sum_{i=1}^{+\infty} f_i z^i$, with $f_1 = 1$ and

$$f_i = \frac{1}{\lambda^i - \lambda} \left[f_i + \sum_{p=2}^{i-1} f_p \sum_{j_1 + \cdots + j_p = i,\, h_{j_k} \geq 1} h_{j_1} \cdots h_{j_p} \right] .$$

To deal with the irrational values of α, let us first consider the linearizable case. Then h is conjugate to an irrational rotation of $\mathrm{Diff}(\mathbb{C}, 0)$. Let C be the set formed by the points of intersection between the leaf of $\widetilde{\mathcal{F}}$ passing by $(1, z_0) \in \mathbb{C}^2$ and the circle $S = \{z \in \Sigma\, ; |z| = z_0\}$. As always, Σ is the transverse section to $\{y = 0\}$ passing by $1 \in \mathbb{C}$. Since an irrational rotation has dense orbits, it follows that C is dense over the circle S. In other words, the leaves are not locally closed, which contradicts Condition 2. In particular, any possible holomorphic first integral f, would be constant everywhere. Indeed, the restriction of f to Σ is constant over circles about $0 \simeq \Sigma \cap \{y = 0\}$. As a result f would be constant everywhere.

We are then reduced to discuss the case in which h is not linearizable. To deal with this case we are going to show the existence of leaves of $\widetilde{\mathcal{F}}$ that are not closed on arbitrarily small neighborhoods of $0 \in \Sigma$.

Let us begin with a simple lemma attributed to J. Lewowicz but, before stating it, we give some definitions that will be useful throughout the text. Let U be a neighborhood of 0, where $h \in \mathrm{Diff}(\mathbb{C}, 0)$ is defined, holomorphic, and injective. Let V be a subset of U and fix a point $x \in V$.

Definition 2.7 *The V-orbit of x is the set of points on V, obtained by iterating h forward (i.e., $h^p = h \circ \cdots \circ h$, p times) and backwards (i.e., $h^{-p} = h^{-1} \circ \cdots \circ h^{-1}$, p times), along with x itself (denoting by $h^0(x) = x$). In other words,*

$$O_V(x) = \{y \in V\, ;\ \ y = h^i(x)\, ,\ i \in \mathbb{Z}\}$$

The *number of iterations* of x is the number of times h is iterated taking x to a point on $O_V(x)$. It is denoted by $\mu_V(x)$ and belongs to $\mathbb{N} \cup \{\infty\}$.

Note that there may exist points x on V such that $\mu_V(x) = \infty$ but $\# O_V(x) < \infty$. These points are called *periodic* on V. Naturally, if $\mu_V(x)$ is finite then $\# O_V(x)$ is necessarily finite.

Definition 2.8 *An element $h \in \mathrm{Diff}(\mathbb{C}, 0)$ is said to have finite orbits if there exists an arbitrarily small open neighborhood V of 0 where h is defined, holomorphic, injective and satisfies*

$$\# O_V(x) < \infty\, ,$$

for every $x \in V$.

Lemma 2.12 *Let K be a compact connected neighborhood of $0 \in \mathbb{R}^n$ and h a homeomorphism of K onto $h(K) \subseteq \mathbb{R}^n$, verifying $h(0) = 0$. Then there exists a point x on the boundary ∂K of K such that the number of iterations in K is infinite, i.e. such that $\mu_K(x) = \infty$.*

Proof. Denote by K° the interior of K and let $\mu_K(x)$ (resp. $\mu_{K^\circ}(x)$) denote the number of iterations of x on K (resp. K°). Suppose, by contradiction, that μ_K only attains finite values on the boundary ∂K of K. Since K is compact, μ_K is uniformly bounded on ∂K, i.e. there exists $N \in \mathbb{N}$ such that

$$\mu_K(x) < N < \infty \quad , \forall x \in \partial K\,.$$

Consider the sets

$$\begin{aligned} A &= \{x \in K \, : \, ; \, \mu_K(x) < N\} \supset \partial K \\ B &= \{x \in K^\circ \, : \, \mu_{K^\circ}(x) \geq N\} \ni 0 \end{aligned}$$

which are non-empty open sets. In fact we can verify that $\partial K \subseteq A$ and $0 \in B$. Moreover, since $\mu_{K^\circ}(x) \leq \mu_K(x)$ for all $x \in K$, those sets are disjoint. Notice that there exists $x_0 \in K$ such that x_0 does not belong to $A \cup B$. Indeed, if it did not exist, then every $x \in K$ would be such that $x \in A \cup B$, in other words, $K = A \cup B$. The connectivity of K would imply that $K = A$ or $K = B$, which is impossible. Thus there exists x_0 such that $\mu_K(x_0) \geq N > \mu_{K^\circ}(x_0)$. It follows that the orbit sets of x_0 on K and on K° are different ($O_K(x_0) \neq O_{K^\circ)}(x_0)$). In other words, the orbit of x_0 passes by ∂K. Let $y \in O_K(x_0) \cap \partial K$, therefore, $\mu_K(y) = \mu_K(x_0) \geq N$, contradicting $\mu_K|_{\partial K} < N$. ■

Lewowicz's Lemma (Lemma 2.12) ensures the existence of points whose orbit never leaves the neighborhood of $0 \in \mathbb{C}$. However, it does not exclude the possibility that all these points are periodic. In this direction, the next proposition will be fundamental.

Proposition 2.5 *If $h \in$ Diff$(\mathbb{C}, 0)$ is not periodic then there exists open neighborhoods U of 0 on $\mathbb{C}$ such that h is holomorphic, injective and for each U, the set of points $x \in U$ with infinite U-orbit is uncountable and 0 is an accumulation point.*

The basic idea behind its proof is to consider sets U_n, for $n \in \mathbb{N}$ such that $U_n = \{x \in U; \, h(x), \cdots, h^n(x)$ are defined, and $h^n(x) = x\}$. If the domain of h^n were connected and $h^n \neq \mathrm{id}$ then each U_n would be a finite set. In particular, $\bigcup_{n \in \mathbb{N}} U_n$ would be countable. By the lemma the set of points with infinite number of iterations is uncountable, hence there would be an uncountable set of points with infinite orbit. The difficulty with this argument is that the domain of h^n may be disconnected and h^n may coincide with the identity on one connected component and not on the component containing 0. This is why the proof of this proposition is slightly more subtle as we need to consider the various connected components of the domain of h^n.

Proof of Proposition 2.5. Let D_{ρ_0} be a closed disc centered at 0 with radius ρ_0. Using the same notations as in Lemma 2.12, we define the following sets:

$$\begin{aligned} P &= \{x \in D_{\rho_0} : \, \mu_{D_{\rho_0}}(x) = \infty, \; \# O_{D_{\rho_0}}(x) < \infty\} \\ F &= \{x \in D_{\rho_0} : \, \mu_{D_{\rho_0}}(x) < \infty, \; \# O_{D_{\rho_0}}(x) < \infty\} \\ I &= \{x \in D_{\rho_0} : \, \mu_{D_{\rho_0}}(x) = \infty, \; \# O_{D_{\rho_0}}(x) = \infty\} \end{aligned}$$

Naturally, $D_{\rho_0} = P \cup F \cup I$. Furthermore, Lemma 2.12 implies that for every $\rho \leq \rho_0$

$$(P \cup I) \cap \partial D_\rho \neq \emptyset\,.$$

Claim: One of the sets, either P or I (or both) is uncountable.

Indeed by Lemma 2.12, for every compact disc D_ρ with $\rho \leq \rho_0$ there is a point x_0 on its boundary with an infinite number of iterations. Hence there are *uncountably* many points x_0 in D_{ρ_0} such that $\mu_{D_{\rho_0}}(x_0) = \infty$. These points belong to $I \cup P$, thus either P or I must be *uncountable*.

Under these notations, the contents of the proposition is that the set I is uncountable. Thus, it suffices to show that if P is uncountable then I is necessarily uncountable as well. This is what we shall do in the sequel.

Before continuing, let us define the following sets containing 0:

$$A_1 = D_{\rho_0}, \quad A_2 = D_{\rho_0} \cap h^{-1}(A_1), \quad \cdots \quad A_n = D_{\rho_0} \cap h^{-1}(A_{n-1}), \quad \cdots.$$

Note that A_n is precisely the domain of definition of h^n.

Let C_n be the connected (compact) component of A_n that contains 0 and let

$$C = \bigcap_{n\in\mathbb{N}} C_n\,.$$

C is, in particular, connected.

Lemma 2.13 *We can assume, without loss of generality, that C is countable.*

Proof. Consider the case where C is uncountable. Suppose for a contradiction that I is countable (and P uncountable). Then $I \cap C$ would also be countable. We consider now $C \cap P$ and note that it must be *uncountable*, otherwise C would be countable, since $C \subset P \cup I$. Let

$$C \cap P = \bigcup_{n\in\mathbb{N}} P_n\,,$$

where P_n is the set of points $x \in C \cap P$ of period n.

Note that there exists a certain $n_0 \in \mathbb{N}$ such that P_{n_0} is infinite, otherwise all of the P_n would be finite and $C \cap P$ would be countable. Being infinite, P_{n_0} has a non-trivial accumulation point in C_{n_0}. The application h^{n_0} is holomorphic on an open neighborhood U_0 of C_{n_0} and it is the identity on $P_{n_0} \cap U_0$. Since this set has an accumulation point on C_{n_0} then $h^{n_0}(z) = z$ on U_0, by the Identity Theorem. By construction, C_{n_0} contains the origin so that U_{n_0} is a neighborhood of $0 \in \mathbb{C}$. This contradicts the hypothesis of non-periodicity of h. Hence I is uncountable. ∎

In view of the preceding lemma, in the sequel we always assume that C consists of countably many points. First, note that there exists $\rho < \rho_0$ such that $C \cap \partial D_\rho = \emptyset$, otherwise C would contain a point x_0 on the boundary of D_ρ for every $\rho < \rho_0$. This is obviously impossible since C countable. From now on let us fix one such $\rho > 0$. We note that, in particular, $C \subseteq D_\rho$.

Next, notice that the sets $C_1 \cap \partial D_\rho, (C_1 \cap C_2) \cap \partial D_\rho, (C_1 \cap C_2 \cap C_3) \cap \partial D_\rho, \ldots$, for ρ as above, form a decreasing sequence of compact sets. Hence the intersection $\bigcap_{n\in\mathbb{N}} C_n \cap \partial D_\rho$ is nonempty, unless there exists $n_0 \in \mathbb{N}$ such that $C_{n_0} \cap \partial D_\rho = \emptyset$. The last case must occur, since ρ was chosen so that $C \cap \partial D_\rho = \emptyset$.

Let K be a compact connected neighborhood of C_{n_0} that does not intersect the other connected components of A_{n_0}, if they exist. In particular one has $\partial K \cap A_{n_0} = \emptyset$.

Lemma 2.14 *For every $x \in \partial K$ there exists $m \leq n_0$ such that $h^m(x)$ is not on D_ρ. Besides $\partial K \cap P = \emptyset$.*

Proof. To verify the existence of m, suppose for a contradiction that for every $m \leq n_0$, $h^m(x)$ belongs to D_ρ, for $x \in \partial K$. In this case, x would have at least n_0 positive iterations of h. This means that $x \in A_{n_0}$, what contradicts the construction of K.

Moreover, if there existed a periodic point of D_{ρ_0} on ∂K, then x would belong to every set A_n. In particular it would belong to A_{n_0}, what is a contradiction. ■

Before continuing to prove that I is uncountable, we define the following sets:

$$\begin{aligned} P' &= \{x \in K;\ \mu_K(x) = \infty,\ \#O_K(x) < \infty\} \\ F' &= \{x \in K;\ \mu_K(x) < \infty,\ \#O_K(x) < \infty\} \\ I' &= \{x \in K;\ \mu_K(x) = \infty,\ \#O_K(x) = \infty\}\,. \end{aligned}$$

Let

$$P' = \bigcup_{n\in\mathbb{N}} P'_n\,,$$

where P'_n is the set of the periodic points on K with period n.

Lemma 2.15 *$P'_n = \mathrm{int}(P'_n) \cup \partial P'_n$, where $\mathrm{int}(P'_n)$ is open without boundary and $\partial P'_n$ is finite, consisting of isolated points.*

Proof. Let p be the limit of a sequence of points in P'_n. Note that we do not assume that this sequence is constituted by pairwise distinct points so that every point belonging to P'_n automatically satisfies this condition. Clearly, it suffices to show that there is a neighborhood of p contained in P'_n in the case that the sequence is not trivial, i.e. p is an accumulation point of the sequence. By definition, there exists an open connected neighborhood W of p where h^n is defined and holomorphic. Moreover h^n is the identity on $P'_n \cap W$. Hence, if p is an accumulation point of the limit sequence of points in P'_n then it $h^n(z) = z$ on W from the Identity Theorem. On the other hand, the K-orbit of p does not intersect ∂K (cf. Lemma 2.14). Thus W can be chosen sufficiently small so that $h(W), h^2(W), \ldots, h^n(W) \subset \mathrm{int}(K)$. This proves that $W \subset \mathrm{int}(P'_n)$. Hence P' consists on the union of isolated points with $\mathrm{int}(P'_n)$. Clearly the number of isolated points must be finite, hence implying the lemma. ■

Lemma 2.16 *The boundary of P' is countable (consisting of the union of the isolated points in P'_n, for $n \in \mathbb{N}$).*

Proof. Let us first consider $\bigcup_{n\in\mathbb{N}} \operatorname{int}(P'_n)$. Naturally it is an open set and we will prove that its boundary is empty. Suppose, for a contradiction, that there exists p in the boundary of $\bigcup_{n\in\mathbb{N}} \operatorname{int}(P'_n)$. Then there is a sequence of points $\{p_k\}$ in $\bigcup_{n\in\mathbb{N}} \operatorname{int}(P'_n)$ converging to p. Fix a small disc about p and choose a point $p_k \in \bigcup_{n\in\mathbb{N}} \operatorname{int}(P'_n)$ belonging to this ball. Hence there is n_0 such that $p_k \in P'_{n_0}$. Clearly p_k belongs to K since $P' \subseteq K$. Furthermore, p belongs to K as well since K is closed. Finally, because K is connected (by construction), there is a path $c : [0,1] \to K$ joining p_k to p (i.e. such that $c(0) = p_k$, $c(1) = p$). Since p_k lies in P'_{n_0} and p does not, there is $t_0 \in [0,1]$ such that $c(t_0)$ belongs to the boundary of $\operatorname{int}(P'_{n_0})$. This is a contradiction to the previous lemma. Thus $\partial P'$ is countable as the union of the finite sets $\partial P'_n$. ■

Finally, we have one last simple lemma that will allow us to conclude the proof.

Lemma 2.17 *F' is an open set of K.*

Proof. Note that, by definition, for points $x \in F^{\textbf{Proof.}}$ we have $\#O_K(x) = \mu_K(x)+1$. Let

$$F' = \bigcup_{n\geq 0} F'_n\,,$$

where F'_n is the set of points in F' such that $\mu_K(x) = n$. We claim that F'_n are open sets. Indeed by continuity of h, every point of F'_n has a neighborhood of points also having n iterations on K. Since F' is the union of open sets, it is itself an open set.
■

To conclude the proposition we proceed as follows. Clearly we have

$$K = \operatorname{int}(P') \cup F' \cup (I' \cup \partial P')\,.$$

We shall analyze separately the following possibilities:

- Suppose that $F' = \emptyset$.

  Then $\partial K \subseteq \operatorname{int}(P') \cup (I' \cup \partial P')$. However, by Lemma 2.14, ∂K does not contain periodic points and so $\partial K \subseteq I' \cup \partial P'$. Now $\partial P'$ is countable, by Lemma 2.16 and by construction ∂K in uncountable. Therefore we conclude that I' is uncountable.

- Suppose that $F' \neq \emptyset$ and $\operatorname{int}(P') = \emptyset$.

  In this case, we have that $K = F' \cup (I' \cup \partial P')$. Note that for sufficiently small values of $r > 0$, the compact disks D_r are contained in K. And so, by Lewowicz's Lemma (Lemma 2.12), we have that $\partial D_r \cap (I' \cup \partial P')$ is non-empty. Therefore, $(I' \cup \partial P')$ is uncountable, and as before, I' must be uncountable as well.

- Suppose that $F' \neq \emptyset$ and $\mathrm{int}(P') \neq \emptyset$.

  Note that $\mathrm{int}(F')$ and $\mathrm{int}(P')$ are non-empty disjoint open sets of $\mathrm{int}(K)$. Thus $K \setminus (\mathrm{int}(F') \cup \mathrm{int}(P'))$ is non-empty so it must be uncountable. Naturally, it is contained on $(I' \cup \partial P')$ and so, I' must also be uncountable.

This completes the proof of Proposition 2.5. ■

Proposition 2.6 *Suppose that the holonomy associated to $\widetilde{\mathcal{F}}$ and to a local transverse section Σ is given by $h = e^{2\pi i\alpha}z + \text{h.o.t.}$, where α is irrational. Then there exist leaves of $\widetilde{\mathcal{F}}$ that are not closed on arbitrarily small neighborhoods of* 0.

Proof. It follows from the preceding proposition that there exists a sequence of points $p_n = h^n(p_0) \in \Sigma$, pairwise distinct and accumulating on a point $p \in \Sigma$. In fact there exists an uncountable number of such sequences. Note that each p_n belongs to a same leaf L. We will show that the leaf L passing through p_n is not closed.

Suppose that the accumulation point p is in L, otherwise the claim is obvious. Suppose that there exists an isolated point q of $L \cap \Sigma$. Consider a path on L connecting these two points, by continuity of the holonomy application, there is also a sequence of points of $L\cap\Sigma$ accumulating on q. In other words, $L\cap\Sigma$ has no isolated points. Therefore, the closure $\overline{L \cap \Sigma}$ of $L \cap \Sigma$ is uncountable (being a perfect set). However, $L \cap \Sigma$ is countable, for every leaf in a foliation can intersect a transverse section only countably many times. Hence, $L \cap \Sigma$ has points of accumulation that do not belong to it. In particular, L is not closed. ■

Proposition 2.7 *Let $\mathcal{F}$ be a foliation satisfying Conditions* 1 *and* 2. *Then its Seidenberg tree only contains singularities that admit a local first integral. In particular, the local holonomy of their separatrizes is finite.*

Proof. Indeed, the preceding discussion allows us to conclude that the only possibility which does not contradict the hypothesis is when $\lambda_1/\lambda_2 = -m/n$ is a rational number. It follows from Theorem 2.17 that the vector field associated to $\mathcal{F}$ is holomorphically conjugate to a linear one, locally given by Equation (2.42). Notice that its solution is given by $\phi(T) = (x_0 e^{mT}, y_0 e^{-nT})$. Hence there exists a local first integral. More precisely, $f(x, y) = x^n y^m$ is constant over the orbits of the vector field. ■

Now we go back to Theorem 2.17 and give an idea of the basic principles behind its proof.

Proof of Theorem 2.17. Suppose that there exists a holomorphic diffeomorphism φ conjugating the holonomies h_1 and h_2 relative to the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively and to a loop S^1 encircling the origin of a separatrix. Let $D(\varepsilon)$ be a disk of radius $\varepsilon > 0$ on a local section Σ transverse to the separatrix (which we can assume to be given, in local coordinates, by $\{y = 0\}$) at the point $(1, 0)$, whose image under h_1 is still contained in the previously chosen neighborhood.

Recall that the vector field X, associated to the blown-up foliation $\widetilde{\mathcal{F}}$ is given by:

$$X = \lambda_1 x(1 + \text{h.o.t.})\partial/\partial x + \lambda_2 y(1 + rmh.o.t.)\partial/\partial y \,.$$

Note that away from $\{x = 0\}$, the leaves of the foliation $\widetilde{\mathcal{F}}$ are transverse to the vertical complex lines, i.e. to the fibers of $\pi_1(x, y) = x$.

First, we define a holomorphic application ϕ on the solid torus $S^1 \times D(\varepsilon)$. This is done by lifting, with respect to π_1 the path γ along S^1 that connects $(1, 0)$ to $(e^{i\theta_0}, 0)$, to a path γ_1 on the leaf L_1 of $\mathcal{F}_1$ that passes by $z \in \Sigma$. Note that this lift is well-defined since the leaves are transverse to the fibers of Π_1. We also lift, with respect to π_1, γ to a path γ_2 on the leaf L_2 of $\mathcal{F}_2$ passing by $\varphi(z)$. Now we extend the diffeomorphism φ by declaring that φ takes the final extremity of γ_1 to the final extremity of γ_2. We have to show that this extension of ϕ is well-defined when γ becomes a loop around $0 \in \{y = 0\}$. This is however clear, since the final extremity of γ_1 (resp. γ_2) is, by construction the image of z by h_1 (resp. h_2) and we have $\varphi \circ h_1 = h_2 \circ \varphi$. Thus we conclude that ϕ is well-defined on $S^1 \times D(\varepsilon)$. ϕ is moreover an extension of φ, i.e. $\phi|_{1 \times D(\varepsilon)} = \varphi$.

Next we wish to extend ϕ holomorphically to a neighborhood of $(0, 0)$ and so that it still conjugates the foliations. Consider them the radial lines R_{θ_0} connecting each point $(e^{i\theta_0}, 0)$ on S^1 to the origin. Let L_1 be the leaf of $\mathcal{F}_1$ passing by $(e^{i\theta_0}, z) \in S^1 \times D(\varepsilon)$ and let γ_{θ_0} be the path over L_1 such that $\Pi(\gamma_{\theta_0}) = R_{\theta_0}$. Analogously, let L_2 be the leaf of $\mathcal{F}_2$ that passes by $\phi(e^{i\theta_0}, z)$ and let η_{θ_0} be the path over L_2 that projects on R_{θ_0} by π_1. We define ϕ to be the application that takes γ_{θ_0} into η_{θ_0} by preserving the projection π_1. By construction, ϕ is a holomorphic conjugation between $\mathcal{F}_1$ and $\mathcal{F}_2$ on its domain. Note that the domain of ϕ is precisely the saturated of Σ by $\mathcal{F}$ which is going to be denoted by $\mathcal{F}_\Sigma$.

However note that, *a priori*, as we follow the radial lines approaching the origin, the union of $\mathcal{F}_\Sigma$ and $\{x = 0\}$ may not contain a neighborhood of $(0, 0)$.

To be more precise, let us revisit the construction of ϕ along the radial lines R_{θ_0}. For example, consider this construction over the radial line R_0 contained in $\{y = 0\}$ and joining the point $(1, 0) = \Sigma \cap \{y = 0\}$ to $(0, 0)$. By definition, we begin with a point $z_0 \in \Sigma$ and consider the path $\gamma_{z_0} : [0, 1] \to L_{z_0}$, $\gamma(0) = z_0$, that lifts only a segment of the radial line R_0 on $\{y = 0\}$, going from $(1, 0)$ to a point $(q, 0)$, with q close to $0 \in \mathbb{C}$. Let Σ_{z_0} be the set consisting of the final extremities, $\gamma_{z_0}(1)$ of the paths γ_{z_0} as above for every $z \in \Sigma$. In particular, Σ_{z_0} is contained in the vertical line $\{x = q\}$. The difficulty to ensure that ϕ will lead to a conjugacy defined around $(0, 0) \in \mathbb{C}^2$ is related to the fact that Σ_{z_0} may not contain a uniform disc about $0 \in \mathbb{C}$ as $q \to 0$.

In his manuscript [M], J.-F. Mattei estimates the size of the sets Σ_{z_0} as above (for all the radial leaves involved and not only the example considered above of R_0). The fundamental result in [M] is the existence of a uniform $\varepsilon > 0$, such that every set Σ_{z_0} contains the ball of radius ε about $0 \in \mathbb{C}$. In particular, he obtains:

Proposition 2.8 (Mattei [M]) *Let $\mathcal{F}$ be a foliation with eigenvalues λ_1 and λ_2, such that* $\mathrm{Re}(\lambda_1/\lambda_2) < 0$*, then $\mathcal{F}_\Sigma \cup \{x = 0\}$ contains a neighborhood of* $(0, 0) \in \mathbb{C}^2$.

The estimate of a uniform $\varepsilon > 0$ as above is done by integrating along R_{θ_0} the differential equation inducing the foliation $\mathcal{F}_1$ (resp. $\mathcal{F}_2$). It is exactly at this point that the assumptions $\lambda_1/\lambda_2 \in \mathbb{R}_-$ plays its role.

This behavior is particularly clear when we consider *real* ODEs with real eigenvalues λ_1 and λ_2 satisfying $\lambda_1/\lambda_2 \in \mathbb{R}_-$. We have that $|y(t)|$ increases as $|x(t)|$ decreases, where $\gamma_{z_0}(t) = (x(t), y(t))$. Indeed this would yield the classical picture of the phase space of a saddle (cf. Figure 2.3).

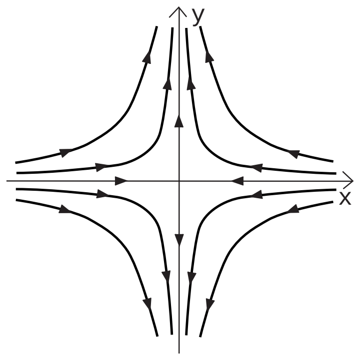

Figure 2.3: Phase space of a saddle.

Note also that, if we consider the case $\lambda_1/\lambda_2 \in \mathbb{R}_+$, the statement would be false. Indeed, with analogous notations, the sets Σ_{z_0} would be contained in balls with radii converging to *zero* when $q \to 0 \simeq (0,0)$.

Although Mattei's manuscript was not published, an extension to higher dimension of this result can be found in [R2] and will be treated in the next section.

Let us now go back to the proof of Theorem 2.17. Whereas ϕ is holomorphic on its domain, this domain clearly does not contain $\{x = 0\}$. To extend ϕ to $\{x = 0\}$ it is however sufficient to note that ϕ is bounded by construction. On the other hand, we already know that ϕ is defined on $U \setminus \{x = 0\}$, where U is a neighborhood of $(0,0) \in \mathbb{C}^2$. Hence the extension of ϕ to $\{x = 0\}$ follows immediately from the Riemann Extension Theorem. Thus the proof of Theorem 2.17 is over. ■

This concludes the second part of the proof of the Mattei-Moussu Theorem.

Part 3: Extension to a Global First Integral

In this part we wish to extend the local first integrals, obtained in the previous section, to a global first integral defined on a neighborhood of the exceptional divisor of the blown-up foliation. Naturally this implies the existence of a first integral of the original foliation, hence yielding Theorem 2.16.

To explain the idea of the construction suppose first that a single blow-up application is sufficient to obtain reduced singularities $p_1, \ldots, p_n$ on the exceptional divisor $E \simeq \mathbb{CP}(1)$. It follows from what precedes that these singularities admit local first integrals. It will also become clear that this case already encloses the main difficulties of the general construction.

The basic idea is to extend the local first integrals around the singularities along the leaves of the blown-up foliation. If it is shown that such an extension is well-defined, then the first integral will be defined on a neighborhood of the exceptional divisor. Indeed, due to Mattei's estimate (Proposition 2.8), the saturated of the local transverse sections Σ_i by the foliation contains a neighborhood of the singularities p_i (except for the separatrizes at each p_i that are transverse to $E = \pi^{-1}(0,0)$). Consequently the extension would be defined on a neighborhood of $\mathbb{CP}(1)$, since they can again be extended over the transverse separatrizes thanks to the Riemann Extension Theorem.

To extend the mentioned first integral, it is necessary to study the *projective holonomy* of $\widetilde{\mathcal{F}}$. Namely, note that $E \setminus \{p_1, \ldots, p_n\}$ is a regular leaf of $\widetilde{\mathcal{F}}$. The holonomy associated to this leaf, which is called the projective holonomy of $\widetilde{\mathcal{F}}$ (or of $\mathcal{F}$), gives us a representation

$$\rho : \Pi_1(E \setminus \{p_1, \ldots, p_n\}) \to \mathrm{Diff}(\mathbb{C}, 0)$$

where $\Pi_1(E \setminus \{p_1, \ldots, p_n\})$ stands for the fundamental group of $E \setminus \{p_1, \ldots, p_n\}$. In the sequel we often make no distinction between the projective holonomy of $\widetilde{\mathcal{F}}$ and the subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$ defined as the image of ρ.

Lemma 2.18 *The projective holonomy of $\widetilde{\mathcal{F}}$ is finite (i.e. $\rho(\Pi_1(E \setminus \{p_1, \ldots, p_n\}))$ is a finite subgroup of* $\mathrm{Diff}(\mathbb{C}, 0)$*).*

Proof. Note that $\rho(\Pi_1(E \setminus \{p_1, \ldots, p_n\}))$ is Abelian. Otherwise we can consider $h \neq \mathrm{id}$ having the form $h = f \circ g \circ f^{-1} \circ g^{-1}$ for $f,\ g \in \rho(\Pi_1(E \setminus \{p_1, \ldots, p_n\}))$. Note however that $h'(0) = f'(0)g'(0)(f'(0))^{-1}(g'(0))^{-1} = 1$, i.e. h is a non-trivial difeomorphism tangent to the identity. Then the local dynamics of h is given by the cardioid-shaped region (cf. Section 2.2.2), and thus there are infinitely many leaves of $\widetilde{\mathcal{F}}$ accumulating on $E = \pi^{-1}(0,0)$. These leaves produce infinitely many leaves of $\mathcal{F}$ accumulating on $(0,0)$ what is impossible.

To complete the proof note that $\Pi_1(E \setminus \{p_1, \ldots, p_n\})$ is generated by small loops around the singularities p_i, $(i = 1, \ldots, n)$. Hence $\rho(\Pi_1(E \setminus \{p_1, \ldots, p_n\}))$ is generated by the local holonomies h_i associated to those singularities. Moreover each h_i has finite order thanks to Proposition 2.5. Since $\rho(\Pi_1(E \setminus \{p_1, \ldots, p_n\}))$ is Abelian, the statement follows. ■

Lemma 2.19 *The group $G = \rho(\Pi_1(E \setminus \{p_1, \ldots, p_n\}))$ is indeed cyclic.*

Proof. Consider the following group:

$$G' = \{f'(0);\ f \in G\}$$

and consider also the homomorphism $\alpha : G(\subset \mathrm{Diff}(\mathbb{C}, 0)) \to G'(\subset \mathbb{C}^*)$ associating to each element f of G its derivative at 0. We claim that the homomorphism is one-to-one and hence a bijection onto its image. Indeed, suppose that f_1 and f_2

are two distinct elements of G such that $\alpha(f_1) = \alpha(f_2)$. Then $f = f_1 \circ f_2^{-1}$ is a non-trivial diffeomorphism tangent to the identity, i.e. such that $f'(0) = 1$. In other words, f has infinite order, which is impossible because the group is finite.

Now let f be an element of G. The derivative $f'(0)$ has modulus equal to 1 since f has finite order. In other words, G' is actually a subgroup of S^1 identified with the complex number of norm 1. A discrete subgroup of S^1 is cyclic: to find a generator it suffices to choose the element of "smallest" argument (different from zero). In particular, G' is cyclic since it is finite. Finally, because α induces a bijection between G and G' we conclude that G is cyclic as desired. ∎

Before continuing, let us work out in detail a simple example of a foliation having a global first integral f. As before we are considering the case where a single blow-up of the foliation gives us singularities p_1, p_2, p_3 in the exceptional divisor E, all in the Siegel domain. They are obtained by the intersection of E with the proper transforms of the separatrizes of $\widetilde{\mathcal{F}}$ (which is simply the set $f^{-1}(0)$). Suppose that the first integral is given by the polynomial

$$f(x,y) = x^{n_1} y^{n_2} (x-y)^{n_3}\,.$$

with n_1, n_2, $n_3 \in \mathbb{N}$.

Note that the vector field X, associated to the foliation $\mathcal{F}$, having f as its first integral is given by

$$X = \frac{\partial f}{\partial y}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial y}$$

Indeed, the 1-form associated to X is $\omega = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ and, since $\omega \wedge df = 0$, we have that f is a first integral of $\mathcal{F}$.

Let us analyze the blown-up foliation on a neighborhood of its singularity given by the intersection of the exceptional divisor $\pi^{-1}(0,0)$ with the proper transform of the separatrix $\{y = 0\}$. In coordinates (x,t), where $\pi(x,t) = (x, tx) = (x,y)$ the vector field is given by $x^{(n_1+n_2+n_3-2)} t^{(n_2-1)} (1-t)^{(n_3-1)} \widetilde{X}$ where

$$\begin{aligned} \widetilde{X} &= (n_2 x(1-t) - n_3 x t)\frac{\partial}{\partial x} + t(-n_1 - n_2 - n_3 + t(n_1+n_2+n_3))\frac{\partial}{\partial t} \\ &= n_2 x[1 + h.o.t]\frac{\partial}{\partial x} - (n_1+n_2+n_3) t[1+h.o.t]\frac{\partial}{\partial t}\,. \end{aligned}$$

Thus the eigenvalues associated to $\widetilde{X}$ on a neighborhood of the singularity p_2 at the separatrix $\{y = 0\}$ are $\lambda_1 = -(n_1+n_2+n_3)$ and $\lambda_2 = n_2$. Recalling the earlier discussion, we conclude that the local holonomy is given by $h_2(z) = e^{-2\pi i n_2/(n_1+n_2+n_3)} z$. Analyzing the other two singularities p_1 at the separatrix $\{x = 0\}$ and p_3 at $\{x = y\}$, by analogous calculations we see that their holonomies are respectively given by $h_1(z) = e^{-2\pi i n_1/(n_1+n_2+n_3)} z$ and $h_3(z) = e^{-2\pi i n_3/(n_1+n_2+n_3)} z$.

The general case of a polynomial first integral given by k separatrizes

$$P(x,y) = x^{n_1} y^{n_2} (x-y)^{n_3} (x - \alpha_4 y)^{n_4} \ldots (x - \alpha_k y)^{n_k}$$

is analogous. At each p_i the holonomy is given by $h_i(z) = e^{-2\pi i n_i/(n_1+\cdots+n_k)}z$, where n_i is the multiplicity of the separatrix that contains p_i. Note that we have fixed 3 separatrizes, i.e., $\{x = 0\}$, $\{y = 0\}$ and $\{x - y = 0\}$. This can always be done without loss of generality, i.e., we may suppose that 3 of the singularities of the blown-up foliation are produced by these separatrizes. In fact, since $\mathrm{PSL}(2, \mathbb{C})$ acts transitively on triples of points in $\mathbb{CP}(1)$, i.e., there always exists a transformation taking p_1, p_2 and p_3 on any other 3 distinct points of $\mathbb{CP}(1)$. Hence the desired normalization can be obtained by a linear change of coordinates.

Naturally, the vector field X having P as its first integral is a homogeneous polynomial of degree $n_1 + \cdots + n_k - 1$. And so, the foliation associated to X is preserved by homotheties. Indeed, one has $X(\lambda x, \lambda y) = \lambda^{(n_1+\cdots+n_k-1)}X(x, y)$, for $\lambda \in \mathbb{C}^*$, so that the "direction" determined by X at the points (x, y) and $(\lambda x, \lambda y)$ is the same. This implies that the holonomy maps h associated to the leaf $E \setminus \{p_1 \ldots, p_k\}$ of $\widetilde{\mathcal{F}}$ (i.e., the elements in the projective holonomy group of $\mathcal{F}$, $\widetilde{\mathcal{F}}$) commutes with homotheties. This can easily be established in details as follows.

Lemma 2.20 *If $X(x, y) = F(x, y)\partial/\partial x + G(x, y)\partial/\partial y$ is homogeneous, then the projective holonomy group h is linear. In other words, the elements of the group $\rho(\Pi_1(\mathbb{CP}(1) \setminus \{p_1, \ldots, p_k\}))$ are linear applications of* $\mathrm{Diff}(\mathbb{C}, 0)$*. In particular the whole group is Abelian.*

Proof. let $\mathcal{F}$ be the foliation associated to X. Since X is homogeneous, the separatrizes of $\mathcal{F}$ are radial lines. In the complement of the separatrizes, the leaves of $\mathcal{F}$ are transverse to the complex vertical lines of $\mathbb{C}$. The blown-up vector field $\widetilde{X}$ is of the form

$$\widetilde{X} = xF(1, t)\frac{\partial}{\partial x} + (G(1, t) - tF(1, t)\frac{\partial}{\partial t}\,, \tag{2.43}$$

We can parameterize the leaves of $\widetilde{\mathcal{F}}$ by $t \to (\varphi(t), t)$. Indeed, away from the proper transforms of the separatrizes of $\mathcal{F}$, $\widetilde{\mathcal{F}}$ is transverse to the Hopf fibration given in these coordinates by $(x, t) \mapsto t$, where t is the natural coordinate along the exceptional divisor. Hence the fact that there is a parametrization for the leaves of $\mathcal{F}$, gives us the differential equation:

$$\varphi'(t) = A(t)\varphi(t)\,, \tag{2.44}$$

where $A(t) = \frac{F(1,t)}{G(1,t)-tF(1,t)}$. Note that the mapping $\varphi_{t_0} : x \mapsto \varphi(t_0, x)$ where t_0 is fixed and x is taken as the initial condition is *linear*. In fact, if φ_1, φ_2 are solutions of (2.44), their sum $\varphi_1 + \varphi_2$ as well as the product $c\varphi_1$, $c \in \mathbb{C}$, are also solutions of (2.44). Hence, $\varphi_{t_0}(x_1, x_2) = \varphi_{t_0}(x_1) + \varphi_{t_0}(x_2)$ and $\varphi_{t_0}(ct_0) = c\varphi_{t_0}(x)$. It follows that $\varphi_{t_0}(x)$ has the form $\lambda(t_0).x$. In particular, the holonomy maps have to be linear. Indeed, the coordinate $\partial/\partial t$ in (2.43) depends only on t, so that holonomy maps actually agree with "time t_0" flows. □

Now going back to our example, we have in addition that the generators $h_1, \ldots, h_k$ are finite. Therefore, the projective holonomy corresponds to the group of rotations

of order $n_1 + \cdots + n_k$. Indeed, they can be chosen as the local holonomies around the singularities of $\widetilde{\mathcal{F}}$.

Let us consider our initial foliation $\mathcal{F}$ satisfying Conditions 1 and 2 of the Mattei-Moussu Theorem. Recall that we are assuming that the singularities of the Seidenberg tree of $\mathcal{F}$ can be reduced by performing a single blow-up. To fix notations, suppose that $\widetilde{\mathcal{F}}$ has k reduced singularities, $p_1, \cdots, p_k$.

Now consider the foliation $\mathcal{F}_P$ that admits a polynomial first integral P, such that $\widetilde{\mathcal{F}}_P$ has precisely the same singularities $p_1, \cdots, p_k$ as $\widetilde{\mathcal{F}}$. Recall that the foliation $\widetilde{\mathcal{F}}_P$ admits a fibration transverse to the exceptional divisor and such that the separatrizes are fibers, which is simply the Hopf fibration, realizing $\widetilde{\mathbb{C}}^2$ as a line bundle over $\mathbb{CP}(1)$ (projection along the proper transforms of the radial lines of $\mathbb{C}^2$).

Naturally we can wonder if $\mathcal{F}$ is actually conjugate to our "model" foliation $\mathcal{F}_P$. If this were the true, then the existence of a first integral for $\mathcal{F}$ would follow as a corollary. To obtain some evidence that this might be the case, let us take the obvious identification of the exceptional divisor E (associated to the blown-up of $\mathcal{F}$) with the exceptional divisor of the blown-up of $\mathcal{F}_P$, which will also be denoted by E. Naturally, $E \setminus \{p_1, \cdots, p_k\}$ is a leaf of the foliation $\widetilde{\mathcal{F}}$. Moreover, recall that the projective holonomy in both cases is cyclic, so that they actually coincide.

The problem of deciding whether or not $\mathcal{F}$, $\mathcal{F}_P$ are conjugate is a special case of the study of moduli spaces for holomorphic foliations. The next theorem is a small piece in this direction.

Theorem 2.18 *Suppose that there is a fibration Π, transverse to the exceptional divisor, and such that the proper transforms of $\mathcal{F}$ are fibers. Then the foliations $\mathcal{F}$ and $\mathcal{F}_P$ are holomorphically conjugate.*

Proof. The assumptions on the existence a transverse fibration on a neighborhood of the origin, allows us to lift paths on $E \setminus \{p_1, \cdots, p_k\}$ to paths in the leaves of $\widetilde{\mathcal{F}}$. The method we use here is essentially the same as the one used in the proof of Theorem 2.17.

Let h be a conjugacy between the projective holonomy groups of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}_P$ represented in Σ. Given a point $z \in \Sigma$ and a path $\gamma \subseteq E \setminus \{p_1, \cdots, p_k\}$ we can lift γ with respect to Π to a path $\tilde{\gamma}$ contained in the leaf of $\widetilde{\mathcal{F}}$ passing by $z \in \Sigma$. Similarly γ can also be lifted with respect to the Hopf fibration into a path contained in a leaf of $\widetilde{\mathcal{F}}_P$.

To construct the conjugacy between $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}_P$ we proceed as follows. Take $z \in \Sigma$ and a path $\gamma \subseteq E \setminus \{p_1, \cdots, p_k\}$ as above. Let $\tilde{\gamma}_1$ be the lift of γ with respect to Π in the leaf of $\widetilde{\mathcal{F}}$ passing by z. Similarly, let $\tilde{\gamma}_2$ be the lift of γ with respect to the Hopf fibration in the leaf of $\widetilde{\mathcal{F}}_P$ passing by $h(z)$. We then impose that H to take the endpoint of $\tilde{\gamma}_1$ to the end of $\tilde{\gamma}_2$. This application is well-defined because the projective holonomies are conjugate. Moreover, thanks to Mattei's estimate (Proposition 2.8), H is defined on the neighborhood of each singularity p_i, except for the separatrizes not contained in the exceptional divisor. As before, we use the Riemann Extension Theorem to define H on the separatrizes. $\square$

Taking this into account, the problem of finding a conjugacy as desired, relies only on the existence of a fibration over $E \setminus \{p_1, \cdots, p_k\}$ having the mentioned separatrizes as fibers. However, the existence of this fibration can only be guaranteed if the number of singularities is at most 4. A counter-example for the case of 5 singularities can be found in [B-M-S]. This prevents us to deduce Theorem 2.16 from a strong result related to the moduli space of foliations. So, in order to treat the general case we cannot rely on the conjugation of $\mathcal{F}$ and $\mathcal{F}_P$. Fortunately, the existence the existence of the fibration is an assumption much stronger than what is really needed to solve the problem of obtaining a first integral.

Indeed, we do not really need an application that conjugates the leaves of $\mathcal{F}$ and $\mathcal{F}_P$, rather, it suffices to find ϕ that is *constant* along the leaves of $\mathcal{F}$ and defined on a neighborhood of E. This is what we shall do in what follows.

Proof of Theorem 2.16 in the case where the foliation is resolved with a single blow-up. From now on we consider the foliation $\mathcal{F}$ along with its blow-up $\widetilde{\mathcal{F}}$. Consider again the transverse section Σ for $\widetilde{\mathcal{F}}$. As it was seen, the projective holonomy of $\widetilde{\mathcal{F}}$ is finite and cyclic. Let z be a local coordinate in Σ in which the generator of the holonomy of $\widetilde{\mathcal{F}}$ is given by

$$z \mapsto e^{2\pi i/m} z\,.$$

Next we consider the function $h : \Sigma \to \mathbb{C}$ given in the above coordinates by $h(z) = z^m$. Clearly h is invariant by the holonomy group of the leaf $E \setminus \{p_1, \cdots, p_k\}$. Now we extend h to a function H by imposing the following,

1. H coincides with h in Σ.

2. H is constant over the leaves of $\widetilde{\mathcal{F}}$ intersecting Σ.

The invariance of h by the projective holonomy group implies that H is well-defined. Now Mattei's estimate ensures that H is defined on a neighborhood of E minus the separatrizes. By Riemann Extension Theorem a continuous extension of H to the separatrizes is automatically holomorphic. Finally we just need to check that H can be continuously extended to the separatrizes setting $H = 0$ over them. This follows immediately from observing that a sequence of leaves L_i of $\widetilde{\mathcal{F}}$ accumulating on the separatrix must accumulate on the exceptional divisor as well. Obviously $h(0) = 0$ so that $H = 0$ over E and the theorem follows. □

Now we are ready to prove the general case of the Theorem of Mattei and Moussu.

Proof of Theorem 2.16. We will show this for the case where 2 blow-ups are sufficient to reduce the singularity. It will become clear that the procedure is the same for any number of blow-ups. Consider a foliation $\mathcal{F}$ such that its Seidenberg tree only contains reduced singularities after two blow-ups. More precisely, suppose that the exceptional divisor E_1 of the blow-up $\widetilde{\mathcal{F}}_1$ by π_1 at $(0,0)$ contains singularities $p_1, \ldots, p_k$, such that the ratio of their eigenvalues is a negative rational number, along with a single degenerated singularity q.

Now we consider the blow-up π_2 of $q \in E_1$, resulting the foliation $\widetilde{\mathcal{F}}_2$. We must show that there is a holomorphic application defined on a neighborhood of the exceptional divisor $E_2 = (\pi_1 \circ \pi_2)^{-1}(0,0)$ of $\widetilde{\mathcal{F}}_2$, which is constant along its leaves.

Denote by E the set $\pi_2^{-1}(p)$ (isomorphic to $\mathbb{CP}(1)$), which only contains reduced singularities, say $q_1, \ldots, q_l$. Consider a local section Σ_1 transverse to E_1 close to p, and the section Σ_2 on E. Consider the "local" (in the sense that it is comes from the singularity p) holonomy h associated to the leaf $E_2 \setminus \{p_1, \cdots, p_k, q_1, \ldots, q_l\}$ and a path with initial and final points, $c(0)$ and $c(1)$ at the intersection of Σ_1 with E_1, and "encircling" E. This group of holonomies is cyclic and finite, since there is a natural correspondence between itself and the projective holonomy group associated to Σ_2 and $E \setminus \{q_1, \ldots, q_l\}$. The last is the cyclic group of rotations as was seen earlier. The other local holonomies associated to loop around each singularity p_i are also finite and cyclic. Thus the "global" holonomy, i.e., the group of holonomies associated to the leaf $E_2 \setminus \{p_1, \cdots, p_k, q_1, \ldots, q_l\}$ of $\widetilde{\mathcal{F}}_2$ and *any* path on this leaf is a cyclic group of rotations. And so we may define the first integral as was done previously and extend it to the neighborhoods of the singularities by Mattei's estimate and Riemann Extension Theorem. □

2.4 Singularities in the Siegel domain

Let us consider two holomorphic foliations $\mathcal{F}_1, \mathcal{F}_2$ on $(\mathbb{C}^2, 0)$ with common non-vanishing eigenvalues λ_1, λ_2 in the Siegel, i.e. satisfying $\lambda_1/\lambda_2 \in \mathbb{R}_-$. There are local coordinates where the foliations have the form given by Equation (2.41). Denote by h_1 (resp. h_2) the holonomy of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) relative to the axis $\{y = 0\}$. According to Theorem 2.17 the foliations $\mathcal{F}_1, \mathcal{F}_2$ are analytically equivalent if and only if the correspondent holonomies are analytically conjugated. This section is devoted to prove a generalization of this ressult for 1-dimensional foliations in the Siegel domain satisfying an open condition and defined on a higher dimensional ambient space.

Let $\mathcal{F}$ be a foliation on $(\mathbb{C}^n, 0)$ and let X be a representative of $\mathcal{F}$ (X is a vector field whose singular set has codimension at least equal to 2). Suppose that the origin is a singular point of $\mathcal{F}$ and denote by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of DX at the correspondent point. Assume that

1. $\mathcal{F}$ has an isolated singularity the origin
2. the singularity of $\mathcal{F}$ is of Siegel type
3. the eigenvalues $\lambda_1, \ldots, \lambda_n$ are all distinct from zero and there exists a straight line through the origin, in the complex plane, separating λ_1 from the other eigenvalues
4. up to a change of coordinates, $X = \sum_{i=1}^{n} \lambda_i x_i (1 + f_i(x)) \partial/\partial x_i$, where $x = (x_1, \ldots, x_n)$ and $f_i(0) = 0$ for all i

On 4. we are assuming the existence of n invariant hyperplanes through the origin.

The analogous of Mattei and Moussu's result for 1-dimensional foliation on higher dimensional spaces can be stated in the following way:

Theorem 2.19 *Let X and Y be two vector fields verifying conditions* 1., 2., 3. *and* 4.. *Denote by h_1^X (resp. h_1^Y) the holonomy of X (resp. Y) relatively to the separatrix of X (resp. Y) tangent to the eigenspace associated to the first eigenvalue. Then h_1^X and h_1^Y are analytically conjugated if and only if the foliations associated to X and Y are analytically equivalent.*

Note that if $\mathcal{F}$ is a foliation on a 3-dimensional space and with an isolated singularity at the origin of strict Siegel type (i.e. such that $0 \in \mathbb{C}$ is contained in the interior of the convex hull of $\{\lambda_1, \ldots, \lambda_n\}$) then conditions 3. and 4. are immediately satisfied (cf. [C]).

The rest of this section is devoted to the proof of this result. The proof of this theorem can be found in either in [E-I] as also in [R2]. Here we follow the approach presented in [R2]. This proof is more detailed than the one presented in the correspondent paper.

The proof of Theorem 2.17 is based on the following fact. Let Σ be a transverse section to one of the separatrizes of $\mathcal{F}$ (as in Theorem 2.17). Then the union of the saturated of Σ by $\mathcal{F}$ with the other separatrix contains a neighborhood of the singular point (cf. Proposition 2.8). In order to prove Theorem 2.19 we shall obtain the correspondent to Proposition 2.8 for foliations on a higher dimensional space:

Proposition 2.9 *Let X be a holomorphic vector field verifying conditions* 1., 2., 3. *and* 4.. *Denote by S the separatrix tangent to the eigenspace associated to λ_1. Then the union of the saturated of a transverse section to S by $\mathcal{F}$ (the foliation induced by X) with the invariant manifold transverse to S contains a neighborhood of the origin.*

Note that the existence of an invariant manifold transverse to the mentioned separatrix is a simple consequence condition 4..

Proof. Assume that X is written in coordinates $(x_1, \ldots, x_n)$ in the form

$$X = \sum_{i=1}^{n} \lambda_i x_i (1 + f_i(x)) \frac{\partial}{\partial x_i} \tag{2.45}$$

Let us normalize the vector fields assuming that $\lambda_1 = 1$.

Note that the x_1-axis corresponds to the separatrix associated to the eigenvalue that can be separated from the others by a straight line through the origin. So fix a positive real constant ε sufficiently close to zero and let $\Sigma = \{(\varepsilon, x_2, \ldots, x_n) : |x_i| \leq \varepsilon, \forall i = 2, \ldots, n\}$ be a transverse section to the x_1-axis at the point $c(0)$, where $c : [0, 2\pi] \to \mathbb{C}^n$ is the curve defined by $c(\theta) = (\varepsilon e^{i\theta}, 0, \ldots, 0)$. Since the domain of c is compact, there exists $0 < \delta < \varepsilon$ such that $\Sigma_\theta = \{(\varepsilon e^{i\theta}, x_2, \ldots, x_n) : |x_i| \leq \delta, \forall i = 2, \ldots, n\}$ is contained in the saturated of Σ.

So let l be, in the complex plane, a straight line through the origin separating λ_1 from the other eigenvalues. Consider the straight line through the origin that is orthogonal to l and denote be L the part of this straight line that is contained in

the left half-plane. Finally, denote by $\bar{L}$ the complex conjugate of L (cf. figure 2.4). Suppose that $v = \alpha + i\beta$, for a certain $\alpha > 0$, is a directional vector of L. Then let

$$T = \{z \in \mathbb{C} : z = x + iy, x \in \bar{L}, -\pi < y \leq \pi\}.$$

It is easy to verify that the image of T by the application map $\phi(z) = \varepsilon e^z$ covers $\{z : |z| \leq \varepsilon\} \setminus \{0\}$. This map is moreover one-to-one.

Fix $z \in T$ and consider the straight line parallel to $\bar{L}$ passing through the fixed point. More specifically we consider the line segment contained in this straight line between its intersection with the imaginary axis and z. This line segment can be parametrized by:

$$c_z(t) = z + \frac{1}{\alpha + i\beta}t \quad \left(= z + \frac{1}{v}t\right)$$

The domain of c_z is the interval $[0, t_z]$ were t_z denotes the instant for which the image of c_z intersects the imaginary axis.

Fix $x_1 \in \mathbb{C}$ such that $|x_1| \leq \varepsilon$ and denote by $z = z(x_1) \in T$ the value for which $\varepsilon e^z = x_1$. Consider the logarithmic spiral curve contained in the x_1-axis given by $r_{x_1}(t) = (\varepsilon e^{c_z(t)}, 0, \ldots, 0)$ for $t \in [0, t_z[$. Note that this curve passes through the point $(x_1, 0, \ldots, 0)$. Denote now by r_x the lift of r_{x_1} to the leaf through the point $x = (x_1, \ldots, x_n)$.

The spiral curve r_{x_1} satisfies $r_{x_1}(0) = (x_1, 0, \ldots, 0)$ and also $|r_{x_1}(t_z)| = \varepsilon$. Consequently, r_x is such that $r_x(0) = x$ and $|p_1(r_x(t_z))| = \varepsilon$, where p_i denoted the projection on the i^{th} component, i.e. $p_i(x) = x_i$. In order to simplify the notation we simply denote by k_x the value of $t_{z(x)}$.

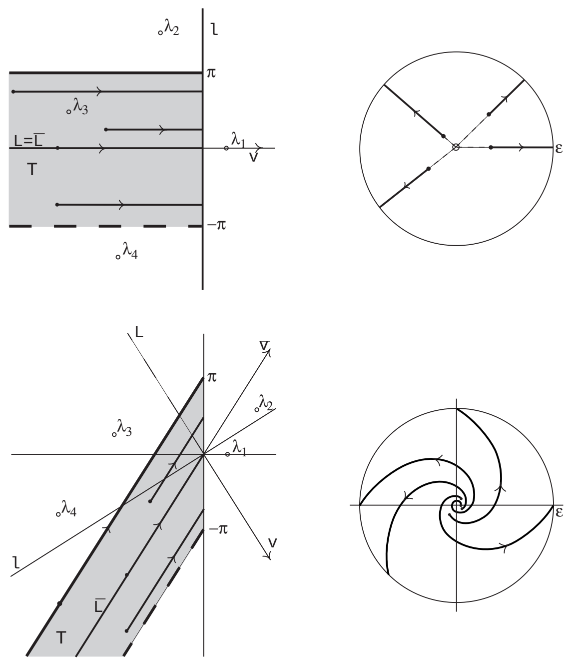

Figure 2.4:

Denote by $\mathfrak{S}_\mathfrak{n}$ the set of vector fields of type (2.45) verifying the conditions 1. to 4.. We have:

Lemma 2.21 *Fix $X \in \mathfrak{S}_n$ and let V denote the set of points x in a small neighborhood of the origin for which r_x intersects Σ_θ, for some $\theta \in [0, 2\pi]$. Then $V \cup \{x_1 = 0\}$ contains a neighbourhood of the origin.*

Proof of Lemma 2.21. Naturally there exists a positive real number $\varepsilon < 1$ for which the projection p_1 is transverse to the leaves of $\mathcal{F}$ on a neighborhood of the polydisk

$$P_{\varepsilon,\delta} = \{x \in \mathbb{C}^n : |x_1| \leq \varepsilon, |x_i| \leq \delta, i \geq 2\}$$

for δ described as above.

Fix $x_1^0 \neq 0$ such that $|x_1^0| \leq \varepsilon$ and denote by z the complex number for which $\varepsilon e^z = x_1^0$. The differential equation associated to X restricted to $x_1 = \varepsilon e^{c_z(t)} = x_1^0 e^{\frac{t}{v}}$ is equivalent to the system of differential equations:

$$\begin{cases} \dfrac{dx_2}{dt} = \dfrac{\lambda_2}{v} x_2(1 + A_2(x_1^0 e^{\frac{t}{v}}, x_2, \ldots, x_n)) \\ \quad\vdots \\ \dfrac{dx_n}{dt} = \dfrac{\lambda_n}{v} x_n(1 + A_n(x_1^0 e^{\frac{t}{v}}, x_2, \ldots, x_n)) \end{cases} \tag{2.46}$$

where A_i are holomorphic functions satisfying $A_i(0) = 0$, for all $i \geq 2$. The constant ε can also be chosen in such a manner that

$$|A_i(x)| \leq \frac{\left|\Re\left(\dfrac{\lambda_i}{v}\right)\right|}{2\left|\dfrac{\lambda_i}{v}\right|}, \quad \forall i \geq 2 \tag{2.47}$$

in $P_{\varepsilon,\delta}$.

Remark 2.5 *Since the eigenvalues $\lambda_2, \ldots, \lambda_n$ are all contained in the same half plane defined by the straight line l and not containing the direction of v it follows that the angle between each one of this eigenvalues and v is greater than $\pi/2$. This implies that λ_i/v has negative real part.*

Fix $x^0 \in P_{\varepsilon,\delta}$ such that all of its components x_i^0 are distinct from zero. It is going to be proved that the solution of the differential equation (2.46) satisfies $x(t) \in P_{\varepsilon,\delta}$ for all $t \in [0, k_{x_1^0}]$.

First of all denote by p the projection on the $n-1$ last components, i.e. in coordinates $x = (x_1, x_2, \ldots, x_n)$ we have $p(x) = (x_2, \ldots, x_n)$. Note that $p(r_{x^0}(t))$ corresponds to the solution of the differential equation (2.46) with initial data $(x_2^0, \ldots, x_n^0)$. Moreover it satisfies $|p_1(r_{x^0}(t))| = \varepsilon$ if and only if $t = k_{x_1^0}$. This is still valid if we restrict ourselves to points x such that $x_i = 0$ for some $i \geq 2$. Note that $\{x_i = 0\}$ is invariant for the foliation.

Integrating the system (2.46) we obtain

$$\operatorname{Log}\left(\frac{x_i(t)}{x_i^0}\right) = \left(\frac{\lambda_i}{v}t + \int_0^t \frac{\lambda_i}{v} A_i(x_1^0 e^{\frac{s}{v}}, x_2(s) \ldots, x_n(s))ds\right)$$

for all $i \geq 2$. Since $\operatorname{Log}(w) = \log|w| + i \arg w$ we have that

$$\log\left|\frac{x_i(t)}{x_i^0}\right| = \Re\left(\frac{\lambda_i}{v}t + \int_0^t \frac{\lambda_i}{v} A_i(x_1^0 e^{\frac{s}{v}}, x_2(s), \ldots, x_n(s))ds\right)$$

and consequently

$$|x_i(t)| = |x_i^0| e^{\Re(\frac{\lambda_i}{v})t + \Re(\frac{\lambda_i}{v}\int_0^t A_i(x_1^0 e^{\frac{s}{v}}, x_2(s), \ldots, x_n(s))ds)}, \quad \forall i \geq 2$$

However

$$\begin{aligned} &\left|\Re\left(\frac{\lambda_i}{v}\int_0^t A_i(x_1^0 e^{\frac{s}{v}}, x_2(s), \ldots, x_n(s))ds\right)\right| \\ &\leq \left|\frac{\lambda_i}{v}\right| \int_0^t |A_i(x_1^0 e^{\frac{s}{v}}, x_2(s), \ldots, x_n(s))|ds \\ &\leq \left|\frac{\lambda_i}{v}\right| \frac{\left|\Re\left(\frac{\lambda_i}{v}\right)\right|}{2\left|\frac{\lambda_i}{v}\right|} t = -\Re\left(\frac{\lambda_i}{v}\right)\frac{t}{2} \end{aligned}$$

and consequently we conclude that

$$\begin{aligned} |x_i(t)| &\leq |x_i^0| e^{\Re(\frac{\lambda_i}{v})t + |\Re(\frac{\lambda_i}{v}\int_0^t A(x_1^0 e^{\frac{s}{v}}, x_2(s), \ldots, x_n(s))ds)|} \\ &\leq |x_i^0| e^{\Re(\frac{\lambda_i}{v})t - \Re(\frac{\lambda_i}{v})\frac{t}{2}} \\ &\leq |x_i^0| \quad (\leq \delta) \end{aligned}$$

for all $t \in [0, k_{x_1^0}]$, since $\Re(\frac{\lambda_i}{v})$ is a negative real number. ■

Condition 4. has been used in the proof of this lemma. Although the condition 4. is always verified by vector fields on $\mathbb{C}^3$ of strict Siegel type, it is not the case for vector fields defined on higher dimensional manifolds (cf. [C]). The condition is also pertinent for vector fields on $\mathbb{C}^3$ whose singular point belongs to the boundary of the Siegel domain. In fact there exist vector fields of that type, verifying 1., 2. and 3. but not admitting any holomorphic manifold tangent to the formal invariant manifold $\{z = 0\}$ (cf. [CM2]).

Remark 2.6 Note that in the case that the angles between λ_1 and the remaining eigenvalues are all greater than $\pi/2$ we have that the spiral curves r_{x_1} are in fact radial.

Let us now finish the proof of Proposition 2.9. Assume that in coordinates $(x_1, \ldots, x_n)$ the vector field X is written in the form (2.45). Let us consider the transverse sections to S defined above and given by $\Sigma_\theta = \{(\varepsilon e^{i\theta}, x_2, \ldots, x_n) : |x_i| \leq \delta, \forall i \geq 2\}$. We have that the saturates of $\Sigma = \Sigma_0$ contains Σ_θ for all $\theta \in [0, 2\pi]$.

Fix a point $x \in P_{\varepsilon,\delta}$ whose first component does not vanish, i.e. such that $x_1 \neq 0$. By Lemma 2.21 the lift of r_x belongs to $P_{\varepsilon,\delta}$ and is such that $r_x(k_{x_1}) \in \Sigma_\theta$, for some θ. So x belongs to the saturated of Σ by $\mathcal{F}$ ■

We are now able to prove the main result of this section.

Proof of Theorem 2.19. Let X, Y be two holomorphic vector fields as stated above and denote by $\mathcal{F}_X$ (resp. $\mathcal{F}_Y$) the foliation associated to X (resp. Y). We can assume that X and Y are written in its normal form (2.45). Denote by $(\lambda_1, \ldots, \lambda_n)$ (resp. $(\beta_1, \ldots, \beta_n)$) the eigenvalues of $DX(0)$ (resp. $DY(0)$). Note that we can assume $\lambda_1 = 1 = \beta_1$ simultaneously. Denote by h^X (resp. h^Y) the holonomy of X (resp. Y) relative to the x_1-axis. It is obvious that if $\mathcal{F}_X$, $\mathcal{F}_Y$ are analytically equivalent then h_X, h_Y are analytically conjugated. So let us assume that h_X, h_Y are analytically conjugated. It will be constructed a holomorphic diffeomorphism defined on a neighborhood of the origin and taking the leaves of $\mathcal{F}_X$ into the leaves of $\mathcal{F}_Y$.

Let $P_{\varepsilon,\delta}$, l and L be as in the proof of Lemma 2.21. We assume that $\varepsilon(< 1)$ is such that Inequality (2.47) is satisfied in $P_{\varepsilon,\delta}$ and also that the projection p_1 is transverse to the leaves of $\mathcal{F}_X \cap P_{\varepsilon,\delta}$ and of $\mathcal{F}_Y \cap P_{\varepsilon,\delta}$ with exception to the ones contained in the invariant manifold $\{x_1 = 0\}$.

Let c be the loop around the singular point defined above ($c(\theta) = (\varepsilon e^{i\theta}, 0, \ldots, 0)$) and let Σ_θ^X, Σ_θ^Y be the "vertical" transverse sections to the common separatrix $\{x_i = 0, i \geq 2\}$ of $\mathcal{F}_X$, $\mathcal{F}_Y$ at the point $c(\theta)$.

Let $h_0 : \Sigma_0^X \to \Sigma_0^Y$ be the analytic diffeomorphism conjugating h_1^X and h_1^Y. Denote by $l_\theta : \Sigma_0^X \to \Sigma_\theta^X$ (resp. $\bar{l}_\theta : \Sigma_0^Y \to \Sigma_\theta^Y$) the applications obtained by lifting the curve c to the leaves of $\mathcal{F}_X$ (resp. $\mathcal{F}_Y$). Consider the holomorphic diffeomorphism $h_t = \bar{l}_t \circ h_0 \circ l_t^{-1}$. Note that this family of diffeomorphisms is well defined in the sense that h_0 coincides with $h_{2\pi}$. In fact the conjugacy relation $\bar{l}_{2\pi} \circ h_0 = h_0 \circ l_{2\pi}$ (note that $l_{2\pi}$, $\bar{l}_{2\pi}$ represent the holomomy map of $\mathcal{F}_X$, $\mathcal{F}_Y$, respectively) together with the fact that $h_{2\pi} = \bar{l}_{2\pi} \circ h_0 \circ l_{2\pi}^{-1}$ implies that $h_{2\pi} = h_0$. Then it has been constructed a diffeomorphism defined on a transverse section to the x_1-axis along c taking the leaves of $\mathcal{F}_X$ into the leaves of $\mathcal{F}_Y$. It remains to extend this diffeomorphism to a neighborhood of the origin. This will be done showing that this diffeomorphism can be transported along the spiral curves r_x.

Denote by r_x^X (resp. r_x^Y) the lift of r_{x_1} to the leaf of $\mathcal{F}_X$ (resp. $\mathcal{F}_Y$) through the point $x = (x_1, \ldots, x_n)$. Lemma 2.21 ensures that for all x in $P_{\varepsilon,\delta}$ there exists $\theta \in [0, 2\pi]$ such that $r_x^X(k_{x_1}) \in \Sigma_\theta^X$ as also $r_x^Y(k_{x_1}) \in \Sigma_\theta^Y$. This is equivalent to saying that the diffeomorphism constructed above can also be transported over the spiral curves r_x^X, r_x^Y. Denote by Φ the resulting diffeomorphism.

An analytic equivalence between $\mathcal{F}_X \setminus \{x_1 = 0\}$ and $\mathcal{F}_Y \setminus \{x_1 = 0\}$ has been constructed in a neighborhood of the origin. Let us now prove that this equivalence can be extended to the invariant manifold $\{x_1 = 0\}$. Since Φ is holomorphic in

$P_{\varepsilon,\delta} \setminus \{x_1 = 0\}$ and since $\{x_1 = 0\}$ is a thin set, it is sufficient to check that Φ is bounded (cf. [G]). In order to do this we need to revisit the construction of Φ.

From now on we use the variables x to $\mathcal{F}_X$ and y to $\mathcal{F}_Y$. Fix a point $x^0 \in P_{\varepsilon,\delta}$ and let $x_1(t) = x_1^0 e^{\frac{t}{v}}$ be a spiral curve in the complex plane identified with the x_1-axis. Let $r^X_{x^0}(t)$ (resp. $r^Y_{x^0}(t)$) be the lift of the spiral curve to the leaf of $\mathcal{F}_X$ (resp. $\mathcal{F}_Y$) through the point x^0. This lift is such that its components are given by

$$
\begin{aligned}
|x_i(t)| &= x_i^0 e^{\int_0^t \frac{\lambda_i}{v}(1+A_i(x_1^0 e^{\frac{t}{v}}, x_2(t),\ldots,x_n(t)))} \\
|y_i(t)| &= y_i^0 e^{\int_0^t \frac{\lambda_i}{v}(1+B_i(x_1^0 e^{\frac{t}{v}}, y_2(t),\ldots,y_n(t)))}
\end{aligned}
$$

for all $2 \le i \le n$ and holomorphic functions A_i, B_i such that $A_i(0) = B_i(0) = 0$. In this way we have that

$$
\frac{y_i(t)}{x_i(t)} = \frac{y_i^0}{x_i^0} e^{\int_0^t \frac{\lambda_i}{v}(B_i - A_i)}
$$

It is now sufficient to prove that $\int_0^t \frac{\lambda_i}{v}(B_i - A_i)$ converges.

On $B_i - A_i$ the function B_i is evaluated at the point $(x, y_2, \ldots, y_n)$ while A_i is evaluated at $(x, x_2, \ldots, x_n)$. Let us sum and subtract $A_i(x, y_2, \ldots, y_n)$ on the expression of $B_i - A_i$. The new function has a Lipshitz component relative to the variables x_i, y_i for $i \ge 2$ (the component $A_i(x, y_2, \ldots, y_n) - A_i(x, x_2, \ldots, x_n)$). The remaining expression can be written as the sum of a holomorphic function F that only depends on x and another one whose Taylor expansion does not have monomials only depending on x. We have to estimate the integral above on each case separately.

Let us begin with the Lipshitz component. We have that

$$
|A_i(x_1, y_2, \ldots, y_n) - A_i(x_1, x_2, \ldots, x_n)| \le \sum_{i=2}^{n} k_i |y_i - x_i| \, .
$$

We first estimate the value of $|y_i(t) - x_i(t)|$ for all $i \ge 2$. Taking account that $|y_i(t) - x_i(t)| \le |y_i(t)| + |x_i(t)|$ we estimate the absolute value of each variable separately. We have

$$
\begin{aligned}
\frac{d}{dt}|x_i(t)|^2 &= 2\Re(\overline{x_i(t)} x_i'(t)) \\
&= 2|x_i(t)|^2 \Re\left(\frac{\lambda_i}{v}(1 + A_i(x_1^0 e^{\frac{t}{v}}, y(t), z(t)))\right)
\end{aligned}
$$

Inequality 2.47 implies that $|\Re\left(\frac{\lambda_1}{v} A_i\right)|$ is less than or equal to $-\frac{\Re(\frac{\lambda_1}{v})}{2}$. In this way

$$
\frac{d}{dt}\log|x_i(t)|^2 \le \Re\left(\frac{\lambda_i}{v}\right) \implies |x_i(k_{x_1^0})| \le |x_i^0| e^{\frac{1}{2}\Re(\frac{\lambda_i}{v})t}
$$

The same inequality can be deduced for $|y_i|$. We conclude therefore that

$$
|A_i(x_1, y_2, \ldots, y_n) - A_i(x_1, x_2, \ldots, x_n)| \le K e^{-\frac{1}{2}\alpha t}
$$

for some positive constants K and α. Let us estimate now the value of the integral of the Lipshitz component. We have

$$\begin{aligned}
&\left|\int_0^t \frac{\lambda_i}{v}[A_i(x_1^0 e^{\frac{s}{v}}, y_2(s), \ldots, y_n(s)) - A_i(x_1^0 e^{\frac{s}{v}}, x_2(s), \ldots, x_n(s))]ds\right| \\
&\leq \int_0^t \left|\frac{\lambda_i}{v}[A_i(x_1^0 e^{\frac{s}{v}}, y_2(s), \ldots, y_n(s)) - A_i(x_1^0 e^{\frac{s}{v}}, x_2(s), \ldots, x_n(s))]\right| ds \\
&\leq \int_0^t K \left|\frac{\lambda_i}{v}\right| e^{-\frac{1}{2}\alpha s} ds \\
&= -\left|\frac{\lambda_i}{v}\right| \frac{2K}{\alpha}\left(e^{-\frac{1}{2}\alpha t} - 1\right)
\end{aligned}$$

and, consequently, the integral converges.

Let us consider now the holomorphic component F depending only on x_1. Suppose that the Taylor's expansion of F around the origin is given by $F(x) = \sum a_j x^j$. Then

$$\begin{aligned}
\int_0^t \frac{\lambda_i}{v} F(x_1^0 e^{\frac{s}{v}}) ds &= \frac{\lambda_i}{v}\int_0^t \sum a_j (x_1^0)^j e^{\frac{js}{v}} ds \\
&= \lambda_i \left(G(x_1^0 e^{\frac{t}{v}}) - G(x_1^0)\right)
\end{aligned}$$

where the Taylor's expansion of G is given by $G(x) = \sum \frac{a_j}{j} x^j$. Since F is holomorphic, so is G. In particular the above integral converges.

Finally, in the remaining case we have that the holomorphic function is bounded by a linear combination of the absolute value of $y_i(t)$, i.e. it is bounded by $\sum_{i=2}^n c_i |y_i|$. This is a simple consequence of the fact that the last component is holomorphic on a neighborhood of the origin and the estimate of the variables y_i. The convergence of the integral in that case follows as in the Lipshitz case.

We have just proved that both Φ^{-1} and Φ are bounded. Thus Φ admits a holomorphic extension $\tilde{\Phi}$ to the invariant manifold $\{x_1 = 0\}$. As $\tilde{\Phi}$ has a holomorphic inverse map (the inverse map is constructed taking now the leaves of Y into the leaves of X in a similar way), $\tilde{\Phi}$ is a diffeomorphism in a neighbourhood of the origin.

It remains be proved that $\tilde{\Phi}$ takes the leaves of $\mathcal{F}_X|_{\{x_1=0\}}$ into the leaves of $\mathcal{F}_Y|_{\{x_1=0\}}$. Let $Z = D\tilde{\Phi}(X \circ \tilde{\Phi})$. As $\mathcal{F}_Y|_{P_{\varepsilon,\delta}\setminus\{x_1=0\}}$ coincides with $\mathcal{F}_Z|_{P_{\varepsilon,\delta}\setminus\{x_1=0\}}$, there exists a holomorphic function f, defined on $P_{\varepsilon,\delta}\setminus\{x_1 = 0\}$, such that $fY = Z$. In particular $f = \frac{Z_2}{Y_2}$, where Y_2 (Z_2) is the second component of Y (Z). As $Y_2 = x_2(\beta_2 + \ldots)$, where dots means terms of order greater than or equal to 1, and both Y_2 and Z_2 are holomorphic, f can be holomorphically extended to $U\setminus\{x_1 = 0, x_2 = 0\}$. Finally, as $\{x_1 = 0, x_2 = 0\}$ is a set of complex codimension greater than 1, f admits a holomorphic extension, $\tilde{f}$, to $\{x_1 = 0, x_2 = 0\}$ [G, pag. 31] and this extension verifies $\tilde{f}Y = Z$ in U. Thus X and Y are analytically equivalent. ■

2.5 Saddle-Node in Higher Dimension

The classification of saddle-nodes in dimension 2 has been discussed in Section 2.3. We shall now approach the case of a saddle-node singularity in higher dimension. In dimension 2 a singularity of saddle-node type has exactly one eigenvalue equal to zero. In higher dimension we can have more eigenvalues equal to zero. We will consider here the case of the so called codimension 1 saddle-nodes, i.e. singularities with exactly one eigenvalue equal to zero.

More specifically, we will consider saddle-node foliations whose representatives belongs up to a linear change of coordinates to $\mathfrak{X}$, the subset of holomorphic vector fields on $(\mathbb{C}^n, 0)$ with an isolated singularity at the origin whose linear part at the singular point has eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $\lambda_1 = 0$ and $0 \notin \mathcal{H}(\lambda_2, \ldots, \lambda_n)$. By $\mathcal{H}(\lambda_2, \ldots, \lambda_n)$ we mean the convex hull of $\{\lambda_i : i = 2, \ldots, n\}$. Furthermore there are no resonance relations between the non-vanishing eigenvalues.

Note that for $n = 3$ the considered vector fields are generic between the vector fields with an isolated singular point that is a codimension 1 saddle-node. It is not difficult to verify that the elements of $\mathfrak{X}$ are analytically equivalent to a vector field of the form

$$Y_p : \begin{cases} \dot{x_1} = x_1^{p+1} \\ \dot{x_i} = \lambda_i x_i + x_1 a_i(x), \quad i = 2, \ldots, n \end{cases} \tag{2.48}$$

where $x = (x_1, \ldots, x_n)$ and where a_i are holomorphic functions such that $a_i(0) = 0$, $\forall i = 2, \ldots, n$. This corresponds to the Dulac's normal form for a codimension 1 saddle-node in $\mathbb{C}^n$.

Let us assume that $p = 1$. The case $p > 1$ can be treated in a similar way but some comments will be made at the end of this section. For simplicity we denote by $Y_{1,\alpha}$ a vector field of the type

$$Y_{1,\alpha} : \begin{cases} \dot{x_1} = x_1^2 \\ \dot{x_i} = x_i(\gamma_i + \alpha_i x_1) + x_1 h_i(x), \quad i = 2, \ldots, n \end{cases}$$

where $\alpha = (\alpha_2, \ldots, \alpha_n) \in \mathbb{R}^{n-1}$ and h_i are holomorphic functions such that $\dfrac{\partial h_i}{\partial x_i}|_0 = 0$, $\forall i = 2, \ldots, n$. We assume further $\gamma_2 = 1$.

Lemma 2.22 *The vector field $Y_{1,\alpha}$ is formally conjugated to*

$$Z_\alpha : \begin{cases} \dot{x_1} = x_1^2 \\ \dot{x_i} = x_i(\gamma_i + \alpha_i x_1), \quad i = 2, \ldots, n \end{cases} \tag{2.49}$$

Summarizing, there exists a formal change of coordinates of the form

$$\hat{H}(x) = (x_1, x_2 + \sum_{k=1}^{\infty} f_{2k}(\bar{x}) x_1^k, \ldots, x_n + \sum_{k=1}^{\infty} f_{nk}(\bar{x}) x_1^k) \tag{2.50}$$

conjugating $Y_{1,\alpha}$ and Z_α. The functions f_{ik} on (2.50) are holomorphic functions on a neighborhood of $0 \in \mathbb{C}^{n-1}$ verifying $f_{i1}(0) = 0$ for all $i \in \{2, \dots, n\}$. Let us denote by $\hat{G}_0$ the set of formal maps of type above.

Although formally conjugated $Y_{1,\alpha}$ and Z_α are not, in general, analytically conjugated. In fact, the formal changes of coordinates presented above are, in general, divergent. Nonetheless $Y_{1,\alpha}$ and Z_α are analytically conjugated by sectors as the next result states. We note that the union of the sectors were $Y_{1,\alpha}$ and Z_α are analytically conjugated constitutes a neighborhood of the origin.

Theorem 2.20 (Theorem of Malmquist) *[Ml, CM1] Let $\hat{H}$ be the unique formal transformation of the form (2.50) conjugating $Y_{1,\alpha}$ and Z_α. Then there exists a holomorphic transformation H defined in $S \times (\mathbb{C}^{n-1}, 0)$, where S a sector with vertex at the origin of $\mathbb{C}$ and angle less then 2π, such that*

a) $dH(Y_{1,\alpha}) = Z_\alpha(H)$, in $S \times (\mathbb{C}^{n-1}, 0)$

b) $H \tilde{\to} \hat{H}$ in S, as $x_1 \to 0$

The union of the sectors satisfying the conditions above constitutes a neighborhood of the origin $0 \in \mathbb{C}$.

A holomorphic map H as above is called a normalizing application. Denoting by S_i the different sectors covering the neighborhood of $0 \in \mathbb{C}$ nd by H_i the correspondent normalizing application we note that $Y_{1,\alpha}$ and Z_α are analytically conjugated if and only if $H_i = H_j$ in $S_i \cap S_j$, $\forall i \neq j$. The description of H_i on the corresponding sector goes as in Section 2.3.

2.5.1 Sectorial Isotropy of the formal normal form

The study of the Sectorial Isotropy for saddle-node foliations as above was considered in [CM1]. In that work the properties of $(H_i \circ H_j^{-1})$ on the overlaps $(S_i \cap S_j) \times (\mathbb{C}^{n-1}, 0)$ are deduced, leading us towards a classifications of codimension 1 saddle-nodes as above.

Let us begin this section with a description of the solutions of the formal normal form. The solutions of (2.49) out of $\{0\} \times (\mathbb{C}^{n-1}, 0)$ are given by

$$x_j(x_1) = c_j x_1^{\alpha_j} e^{-\frac{\gamma_j}{x}}, \quad j \geq 2$$

where $(c_2, \dots, c_n) \in \mathbb{C}^{n-1}$. Basically they are parameterized by the constants $(c_2, \dots, c_n)$. The main objective in this subsection is to relate the solutions of Z_α with the solutions of $Y_{1,\alpha}$ on each sector given by the Theorem of Malmquist.

Denote by φ_i, for $i = 2, \dots, n$, the argument of the eigenvalue γ_i and let $x_1 = re^{i\theta}$. Since

$$x_j(re^{i\theta}) = c_j r^{\alpha_j} e^{i\theta\alpha_j} e^{-\frac{|\gamma_j|}{r}(\cos(\varphi_j - \theta) + i\sin(\varphi_j - \theta))}$$

for a fixed θ, the behavior of $x_j(x_1)$ along the curve $x_1 = re^{i\theta}$ as $r \to 0$ is given by the term $\frac{|\gamma_j|}{r}\cos(\varphi_j - \theta)$. More specifically, if $\cos(\varphi_j - \theta) > 0$ (resp. $\cos(\varphi_j - \theta) < 0$) then $x_j(re^{i\theta})$ goes to zero (resp. infinity) as r goes to zero.

A sector where $(x_2(x_1), \dots, x_n(x_1)) \to (0, \dots, 0)$ as $r \to 0$ is called an attractor sector. This kind of sector is constituted by the directions θ for which $\cos(\varphi_j - \theta) > 0$, $\forall j = 2, \dots, n$. A sector where $\cos(\varphi_j - \theta) < 0$, $\forall j = 2, \dots, n$, is called a saddle sector (in this case $|x_j(x_1)| \to \infty$, $\forall j = 2, \dots, n$). Contrary to the case of saddle-nodes in $\mathbb{C}^2$ there exists, in general, sectors that are neither attractors nor saddles. Those sectors are called mixed and are characterized by the condition $\cos(\varphi_i - \theta)\cos(\varphi_j - \theta) < 0$, for some $i \neq j$. In fact mixed sectors does not exists only in the case that $\gamma_i \in \mathbb{R}$ (or $\mathbb{R}^+$ since we are assuming $\gamma_2, \dots, \gamma_n$ in the Poincaré domain).

The directions for which there exists j such that $\cos(\varphi_j - \theta) = 0$ are called singular directions of the solution. They are given by $\theta = \varphi_j \pm \frac{\pi}{2}$, $j = 2, \dots, n$. For simplicity we sometimes say that $\theta \in S$ instead of that $x = re^{i\theta} \in S$.

2.5.2 The sectors where the Theorem of Malmquist is valid

Let S be a sector of $\mathbb{C}$ with vertex at the origin and angle less then 2π. Denote by $\Lambda_{Z_\alpha}(S)$ the group of holomorphic transformations $H : S \times (\mathbb{C}^{n-1}, 0) \to S \times (\mathbb{C}^{n-1}, 0)$ verifying:

a) $dH(Z_\alpha) = Z_\alpha(H)$

b) H is asymptotic to the identity Id as $x_1 \to 0$, $x_1 \in S$

$\Lambda_{Z_\alpha}(S)$ is a presheaf and we denote by Λ_{Z_α} the sheaf associated to the given presheaf.

An element H tangent to the identity and preserving the first component can be written in the form

$$\begin{aligned} H(x) = (x_1, x_2 + a_{20}(x_1) + &\sum_{|Q|\geq 1} a_{2Q}(x_1)\bar{x}^Q, \dots, \\ & x_n + a_{n0}(x_1) + \sum_{|Q|\geq 1} a_{nQ}(x_1)\bar{x}^Q) \end{aligned} \tag{2.51}$$

where $Q = (q_2, \dots, q_n)$, $|Q| = q_2 + \dots + q_n$ and $\bar{x}^Q = x_2^{q_2} \dots x_n^{q_n}$. We look for conditions under which $H \in \Lambda_{Z_\alpha}(S)$. In order to do that we need to interpret the conditions a) and b) above.

Condition a) expresses that the leaves of $Z_\alpha|_{S\times(\mathbb{C}^{n-1},0)}$ are preserved by the elements of $\Lambda_{Z_\alpha}(S)$. This implies that

$$a_{jQ}(x_1) = a_{jQ} x_1^{-((Q,\alpha)-\alpha_j)} e^{\frac{(Q,\gamma)-\gamma_j}{x_1}} \tag{2.52}$$

where $\gamma = (\gamma_2, \dots, \gamma_n)$.

Condition b) says that $H \tilde{\to} Id$ as x_1 goes to 0, with $x_1 \in S$. This is simply equivalent to say that $a_{jQ}(x_1) \tilde{\to} 0$ as x_1 goes to 0, with $x_1 \in S$, for all $j \in \{2, \dots, n\}$ and for all $Q \in \mathbb{N}_0^{n-1}$.

Denote by φ_{jQ} the argument of the complex number $(Q,\gamma) - \gamma_j$ and let $x_1 = re^{i\theta}$. For a fixed θ the behavior of $a_{jQ}(x_1)$ as $r \to 0$, is given by $\cos(\varphi_{jQ} - \theta)$. The directions θ for which $\cos(\varphi_{jQ} - \theta) = 0$ for some j and Q, are called singular directions of the sheaf Λ_{Z_α} and are given by $\theta_{jQ}^{\pm} = \varphi_{jQ} \pm \frac{\pi}{2}$, $j = 2, \dots, n$. We note that the singular directions of the solution are singular directions of the sheaf either. In fact the eigenvalue γ_j corresponds to the complex number φ_{j0}.

In order to study the behavior of the arguments of $(Q,\gamma) - \gamma_j$, for $Q \in \mathbb{N}_0^{n-1}$, we shall represent all these numbers in the complex plane (see figure 2.5). We note that the singular directions of the sheaf are dense in the mixed sectors, while they are discrete in the attractor and saddle ones. Although discrete in the attractor and saddle sectors they accumulate on the singular directions of the solution.

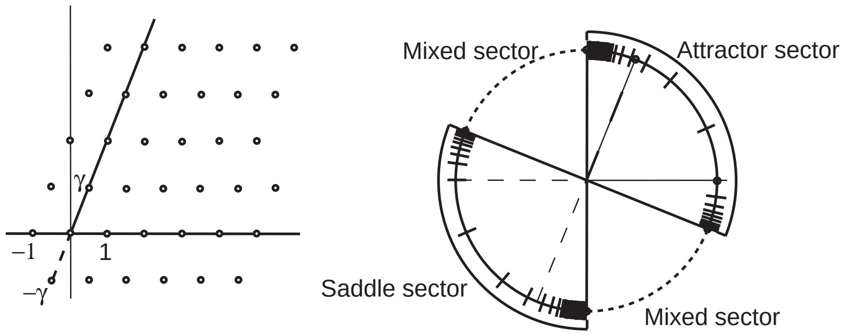

Figure 2.5: On the left the complex numbers $(Q,\gamma) - \gamma_j$. On the right the singular directions of the sheaf Λ_{Z_α} which are dense in the mixed sectors.

Let us now describe how to construct the sectors where the Theorem of Malmquist is valid. First of all let us consider a direction φ_0 in the attractor sector that is not a singular direction of the sheaf Λ_{Z_α}. The sectors where the Theorem of Malmquist is valid are the sectors obtained by extending the sectors between the angles φ_0 and $\varphi_0 \pm \pi$ till reach a singular direction of the sheaf Λ_{Z_α} (see (figure 2.6)). Since singular direction are discrete on the attracting and saddle sectors, both sectors are well defined and have amplitude greater than π. Denote each one of this sectors by S_1 and S_2.

By definition $S_1 \cap S_2$ is the union of two open sets, S_+ and S_-, contained in the attractor sector and in the saddle sector respectively. In particular $S_+ \cap S_- = \emptyset$. The saddle and attractor sectors, denoted by S_s and A_s respectively, are antipodes, i.e. $S_s = A_s + \pi = \{e^{\pi i}a : a \in A_s\}$. The sets S_+ and S_- are also antipodes.

2.5.3 The importance of the pre-sheaves $\Lambda_{Z_\alpha}(S_+)$ and $\Lambda_{Z_\alpha}(S_-)$

Let $\hat{H}$ be the unique formal diffeomorphism of $\mathbb{C}\{\bar{x}\}[[x_1]]$ conjugating $Y_{1,\alpha}$ and Z_α.

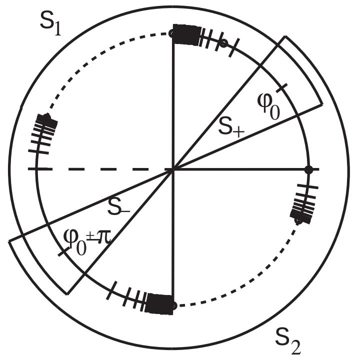

Figure 2.6: How to construct sectors where the Theorem of Malmquist is valid

Proposition 2.10 *Let S_1 and S_2 be the sectors where the Theorem of Malmquist is valid. Denote by H_1 (resp. H_2) the normalizing application defined on S_1 (resp. S_2). Then, $H_j \circ H_i^{-1}|_{S_+}$ (resp. $H_j \circ H_i^{-1}|_{S_-}$) belongs to $\Lambda_{Z_\alpha}(S_+)$ (resp. $\Lambda_{Z_\alpha}(S_-)$).*

Let g_+ (resp. g_-) be the restriction of $H_2 \circ H_1^{-1}$ to S_+ (resp. S_-). We have that $Y_{1,\alpha}$ is analytically conjugated to its formal normal form Z_α if and only if H_1 coincides with H_2 on the overlaps, i.e. if and only if both g_+ and g_- reduces to the identity. The diffeomorphisms g_+, g_- represents the gluing of the leaves on the overlaps. As already mentioned they take the form 2.51 with $a_{jQ}(x_1) = a_{jQ} x_1^{-((Q,\alpha)-\alpha_j)} e^{\frac{(Q,\gamma)-\gamma_j}{x_1}}$ and $a_{jQ}(x_1)$ asymptotic to zero as x_1 goes to zero.

2.5.4 Gluing of the leaves

In order to describe the gluing of the leaves on the overlaps, it is important to obtain a description of the maps of the form (2.51) on S_+ and S_-. In other words, we shall obtain a description of the sheaves $\Lambda_{Z_\alpha}(S_+)$ and $\Lambda_{Z_\alpha}(S_-)$. The next two propositions describe their fundamental properties:

Proposition 2.11 *Fix a sector S and assume that $a_{jQ} \neq 0$ on S. Then $\cos(\varphi_{jQ} - \theta) < 0$, for all θ such that $re^{i\theta} \in S$.*

Proof. Since the elements of $\Lambda_{Z_\alpha}(S)$ are asymptotic to the Identity it follows that $a_{jQ}(x_1) \overset{\sim}{\longrightarrow} 0$ as x_1 goes to 0. Let $x_1 = re^{i\theta}$. For a fixed θ the behavior of $a_{jQ}(x_1)$, as r goes to 0, is given by the real part of $\frac{(Q,\gamma)-\gamma_j}{x_1}$, i.e. by $\dfrac{|(Q,\gamma)-\gamma_j|}{r}\cos(\varphi_{jQ} - \theta)$.

Suppose that $a_{jQ} \neq 0$. Assume for a contradiction that there exists $\theta \in S$ such that $\cos(\varphi_{jQ} - \theta) > 0$. Then

$$\frac{|(Q,\gamma)-\gamma_j|}{r}\cos(\varphi_{jQ} - \theta) \overset{r\to 0}{\longrightarrow} +\infty$$

which means that $a_{jQ}(x_1)$ is not asymptotic to the zero function, contradicting our assumption. ■

Proposition 2.12 *There exists a duality between* $\Lambda_{Z_\alpha}(S)$ *and* $\Lambda_{Z_\alpha}(S+\pi)$ *in the following sense: if* $a_{jQ} \neq 0$ *in* S *then* $a_{jQ} = 0$ *in* $S+\pi$. *Note that by* $S+\pi$ *we mean the antipode of* S.

In particular there exists a duality between $\Lambda_{Z_\alpha}(S_+)$ and $\Lambda_{Z_\alpha}(S_-)$. Therefore we just need to know the constants that can be non zero in $\Lambda_{Z_\alpha}(S_+)$, i.e. we need to know for which pair (j, Q) we have $\cos(\varphi_{jQ} - \theta) < 0$, for all θ such that $re^{i\theta} \in S_+$ ($r \in \mathbb{R}^+$).

The next result expresses how the gluing of the leaves is done.

Proposition 2.13 *Fix a sector* S *and take an element* H *of the presheaf* $\Lambda_{Z_\alpha}(S)$. *Thus* H *transforms the solution of the differential equation associated to the formal normal form given by:*

$$x_j(x_1) = c_j x_1^{\alpha_j} e^{-\frac{\gamma_j}{x_1}}, \quad j \geq 2$$

into the solution of the same equation given by:

$$x_j(x_1) = (c_j + a_{j0} + \sum_{|Q|\geq 1} a_{jQ} c^Q) x_1^{\alpha_j} e^{-\frac{\gamma_j}{x_1}}, \quad j \geq 2$$

where $c^Q = c_2^{q_2} \dots c_n^{q_n}$.

Fixed a sector S, each $c = (c_2, \dots, c_n) \in (\mathbb{C}^{n-1}, 0)$ represents a leaf of the foliation associated to the restriction of Z_α to $S \times (\mathbb{C}^{n-1}, 0)$. More specifically, c works like a parametrization of the leaves in S. If S does not contain singular directions of the sheaf, we can identify $\Lambda_{Z_\alpha}(S)$ with the set of transformations in the space of the leaves given by

$$\{c \mapsto (c_2 + a_{20} + \sum_{|Q|\geq 1} a_{2Q} c^Q, \dots, c_n + a_{n0} + \sum_{|Q|\geq 1} a_{nQ} c^Q)\}$$

We also denote this set by $\Lambda_{Z_\alpha}(S)$. The presheaf $\Lambda_{Z_\alpha}(S)$ expresses then that the leaf of $Z_\alpha|_{S\times(\mathbb{C}^{n-1},0)}$ parameterized by $(c_2, \dots, c_n)$ is taken into the leaf parameterized by $(c_2 + a_{20} + \sum_{|Q|\geq 1} a_{2Q} c^Q, \dots, c_n + a_{n0} + \sum_{|Q|\geq 1} a_{nQ} c^Q)$. Since S_+ (resp. S_-) does not contain singular direction of the sheaf the interpretation above can be given to the presheaf $\Lambda_{Z_\alpha}(S_+)$ (resp. $\Lambda_{Z_\alpha}(S_-)$).

Let us now explain how $\Lambda_{Z_\alpha}(S_+)$ can be determined for a given sector S_+. Note that S_1, S_2 are not uniquely determined and so do S_+, S_-. The first step is to choose sectors S_1 and S_2 or, equivalently, sectors S_+ and S_- we are going to work on. Then we consider the set of complex numbers $C = \{(Q, \gamma) - \gamma_i : i = 2, \dots, n, Q \in \mathbb{N}_0^{n-1}\}$. We can assume without loss of generality that $0 = \arg(\gamma_2) \leq \dots \leq \arg(\gamma_n) < \pi$.

Denote by K the sector, with vertex at the origin, whose elements have arguments between $0 = \arg(\gamma_2)$ and $\arg(\gamma_n)$. Then we choose two directions not contained in K such that the two of the four sectors defined by those directions do not contain any element of C in its interior (figure 2.7). Fix one of those two sectors and denote it by S. If $S + \frac{\pi}{2}$ is contained in the attractor sector we take $S_+ = S + \frac{\pi}{2}$ otherwise

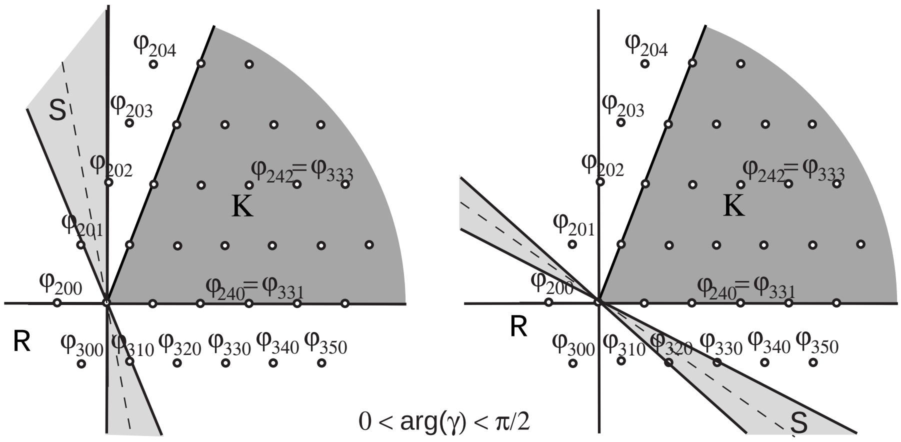

Figure 2.7: Each point represents an element of the form $(Q,\gamma)-\gamma_i$ for some $i=2,\dots,n$ and $Q\in\mathbb{N}^{n-1}$

we take $S_+ = S - \frac{\pi}{2}$. Note that S can be chosen so close to the real axis as we want since we can chose the two directions above so close to π as we want.

The constants a_{jQ} that can be non zero in $\Lambda_{Z_\alpha}(S_+)$ are those for which $(Q,\gamma)-\gamma_j$ is in the half plane, defined by the bisectrix of S, not containing S_+. Denote this region by R. For example, if we look at figure 2.7, on the left case $\Lambda_{Z_\alpha}(S_+)$ is given by

$$\{(y,z)\mapsto(y+a_{200}+a_{201}z, z+a_{300})\}$$

while on the right one $\Lambda_{Z_\alpha}(S_+)$ is given by

$$\{(y,z)\mapsto(y+a_{200}, z+a_{300}+a_{310}y+a_{320}y^2)\}.$$

More specifically, the monomial coefficient of c^Q on the $(i-1)^{th}$-component of g_+ can be non-zero if and only if $(Q,\gamma)-\gamma_i\in R$. We should note that the elements of $\Lambda_{Z_\alpha}(S_+)$ are always polynomial while the elements $\Lambda_{Z_\alpha}(S_-)$ are always tangent to the identity, i.e, a_{i0} must be equal to 0 in S_- for all $i=2,\dots,n$.

In the case that $0=\arg(\gamma_2)=\ldots=\arg(\gamma_n)$ the constants a_{jQ} that can be non zero in $\Lambda_{Z_\alpha}(S_+)$ are those such that $\arg((Q,\gamma)-\gamma_j)=\pi$. In particular, we have only one element Q such that $\arg((Q,\gamma)-\gamma_j)=\pi$ in the case of a saddle-node in dimension 2. Therefore the element on S_+ is just a translation as already mentioned.

Bibliography

[Ak] D. Akhiezer, *Lie group actions in complex analysis. Aspects of mathematics*, E27. Friedr. Vieweg & Sohn, Braunschweig, 1995.

[A] V. Arnold, *Chapitres supplémentaires de la théorie des équations différentielles ordinaires*, "Mir", Moscow (1984)

[Be] A. Beauville, *Complex algebraic surfaces*, London Math. Soc. Student Texts, 34, Cambridge University Press (1996)

[B-M-S] M. Berthier, R. Meziani, P. Sad, *On the classification of nilpotent singularities*, Bull. Sci. Math. 123 (1999), no. 5, 351-370.

[B-Mo] S. Bochner, D. Montgomery, *Groups actions on analytic manifolds*, Ann. of Math. (2) 48 (1947), p. 659-669.

[C] C. Camacho, N. Kuiper, J. Palis, *The topology of holomorphic flows with singularity*, Publ. Math., Inst. Hautes Étud. Sci., 48, 5-38 (1978).

[CM1] J.C. Canille Martins, *Contribuição ao estudo local de fluxos holomorfos com ressonância*, PhD Thesis, IMPA-CNPq, Brasil (1985)

[CM2] J. C. Canille Martins, *Holomorphic flows in* $\mathbb{C}^3, 0$ *with resonances*, Trans. Amer. Math. Soc. 329, 825-837 (1992).

[Car-G] L. Carleson, T.W. Gamelin *Complex Dynamics*, New York: Springer-Verlag, (1993) p. 174.

[Ca-Se] H. Cartan, J-P. Serre, *Un théorème de finitude concernant les variétés analytiques compactes*, C. R. Acad. Sci. Paris 237, (1953), 128-130.

[D] B. Deroin, *Sur la conjecture de Remmert*, Bol. Soc. Mat. Mexicana (3) 9, no. 2 (2003), 213234.

[E-I] P. M. Elizarov, Yu S. Il'Yashenko, *Remarks in the orbital analytic classification of germs of vector fields*, Math. USSR, Sb., Vol. 49, No. 1, 111-124 (1984).

[F-Gr] K. Fritzsche, H. Grauert, *From Holomorphic Functions to Complex Manifolds*, Sringer (2000).

[Fu] W. Fulton, *Algebraic Curves: an Introfuction to Algebraic Geometry*, Addison-Wesley Pub. (1989).

[G-L] E. Ghys, J. Leys, *Lorenz and modular flow: a visual introduction*, Feature Column, AMS, Novembre 2006.

[G-R] E. Ghys, J.-C. Rebelo, *Singularités des Flots Holomorphes II*, Ann. Inst. Fourier, Vol. 47, No. 4, 1117-1174 (1997).

[G] P.A. Griffiths, *Introduction to Algebraic Curves*, Amer. Math. Soc. (1989) p. 225.

[G-Ha] P.A. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Wiley Classics Library. John Wilet & Sons Inc., New York (1994).

[Gu] A. Guillot, *Sur les équations d'Halphen et les actions de* $\mathrm{SL}_2(\mathbf{C})$, Publ. Math. Inst. Hautes Études Sci. No. 105 (2007), 221294.

[H-K-M] H. Hukuara, T. Kimura, T. Matuda, *Équations diférentielles ordinaires du premier ordre dans le champ complexe*, Publ. Math. Soc. of Japan, 1961.

[Ko] A. Kosinski, *Differential manifolds*, Dover (2007)

[L] E.N. Lorenz, *Deterministic nonperiodic flow*, J. Atmosph. Sci., 20 (1963), 130-141

[Ml] J. Malmquist, *Sur l'étude analytique des solutions d'un système d'équations dans le voisinage d' un point de indetermination*, Acta Math. 73 (1941), 87-129

[Mr] G. Marino, *Equações Diferenciais Ordinárias Complexas*, Master's Thesis, PUC-Rio, Brasil (2007)

[Ma-Ro] F. Martin, E. Royer, *Formes Modulaires et Périodes. Formes modulaires et transcendance*, Sémin. Congr., 12, Soc. Math. France, Paris (2005)

[Ma-R] J. Martinet, J.-P. Ramis, *Problèmes de modules pour des équations différentielles non linéaires du premier ordre*, Publ. Math. I.H.E.S., **55**, (1982), p.63-164.

[M-M] J.-F. Mattei, R. Moussu, *Holonomie et intégrales premières*, Ann. Sc. E.N.S., IV Sér. **13** (1980), p. 469-523.

[M] J.-F. Mattei, unpublished manuscript (1996).

[Mu] D. Mumford, *Algebraic Geometry I, Complex Projective Varieties*, Springer (1976).

[PM] R. Pérez-Marco, *Solution Complète au Problème de Siegel de Linéarisation d'une Application Holomorphe au Voisinage d'un Point Fixe (D'après J.-C. Yoccoz)*, Sém. Bourbaki, Astérisque 206 (1992), p.273-310.

[Re] J.-C. Rebelo, Lecture notes of a course given at Stony Brook during fall 2001.

[R1] H. Reis *Semi-complete vector fields of saddle-node type in* $\mathbb{C}^n$, Trans. Amer. Math. Soc. **360**, (2008), 12, 6611-6630.

[R2] H. Reis *Equivalence and semi-completude of foliations*, Nonlinear Analysis, theorey, Methods and Applications, Vol. 64, No. 8, 1647-1896 (2006).

[Sd] A. Seidenberg, *Reduction of Singularities of the Differentiable Equation* $Ady = Bdx$, Amer. J. Math., 90 (1968), p. 248-269.

[Se] J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier, Grenoble 6 (1955-1956), 142.

[Sh1] I. Shafarevich, *Basic Algebraic Geometry 1*, Springer (1994).

[Sh2] I. Shafarevich, *Basic Algebraic Geometry 2*, Springer (1994).

[T] W. Tucker, *A rigorous ODE solver and Smales 14th problem*, Found.Comput. Math. 2, no 1 (2002), 53117.

[V] M. Viana, *What's new on Lorenz strange attractors*, Math. Intelligencer 22-3 (2000), 6-19.

[Vo] S. Voronin, *Analytic Classification of Germs of Conformal Mappings* $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ *with Identity Linear Part*, Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, vol. 15, No.1 (1981) p. 1-17.